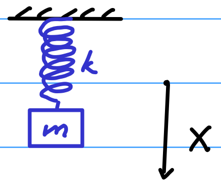
Svingninger



Eks: Fjær og masse.

Fjærkraft: $-kX$

$mg - kX = m\ddot{X}$. Likevekt (i ro): $mg = kX_0 \Rightarrow X_0 = \frac{mg}{k}$

Def.: $x = X - X_0$ (avstand fra likevekt), $\ddot{x} = \ddot{X}$.

$mg - kX = mg - k(x + X_0) = mg - kX_0 - kx = -kx$

Gir: $\ddot{x} = -\frac{k}{m}x$ eller $\ddot{x} = -\omega^2 x$, der $\omega = \sqrt{\frac{k}{m}}$.

Løsning: $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, $e^{i\omega t} = \cos\omega t + i\sin\omega t$ (Euler)

$x(t) = C_1\cos\omega t + C_2\cos\omega t + iC_1\sin\omega t + iC_2\sin(-\omega t)$

$= (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t$

$= B_1\cos\omega t + B_2\sin\omega t$, $B_1 = C_1 + C_2$, $B_2 = i(C_1 - C_2)$

Kan også skrives $x(t) = D\cos(\omega t - \delta)$, for konst. D og δ (ukesoppgave)

Altså: Svingninger!

Kompleks representasjon: $x(t) = D\,\mathrm{Re}\, e^{i(\omega t - \delta)}$

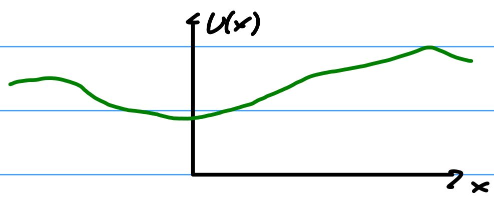
$= \mathrm{Re}\, De^{i(\omega t - \delta)}$

$= \mathrm{Re}\, De^{-i\delta}e^{i\omega t}$

$= \mathrm{Re}\, Ce^{i\omega t}$, C kompleks konst.

Fordel: Lettere å addere ...

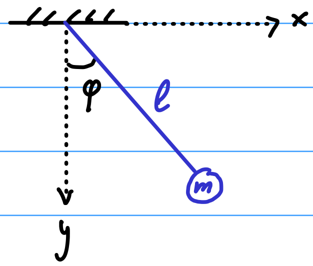
Mer generelt: Potensiell energi $U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \ldots$



For små $|x|$: $m\ddot{x} = -\frac{dU}{dx} = \cancel{U'(0)} + \underbrace{U''(0)}_{\text{effektivt en fjærkonst.}}x \Rightarrow$ harm. osc.

Eks: Pendel

$$\mathcal{L} = \frac{1}{2} m \ell^2 \dot{\varphi}^2 - mg\ell(1-\cos\varphi)$$

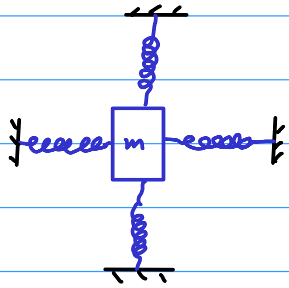


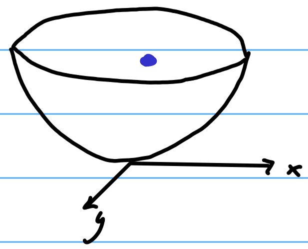
Lagrange: $\dfrac{\partial \mathcal{L}}{\partial \varphi} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right)$

$$\Rightarrow \; -mg\ell \sin\varphi = \frac{d}{dt} m\ell^2 \dot{\varphi} = m\ell^2 \ddot{\varphi}$$

$$\Rightarrow \; \ddot{\varphi} = -\frac{g}{\ell}\sin\varphi = -\frac{g}{\ell}\left(\varphi - \frac{\varphi^3}{3!} + \dots\right) \approx -\frac{g}{\ell}\varphi = -\omega^2\varphi, \quad \omega = \sqrt{\frac{g}{\ell}}$$

Eks: 2D-oscillator:





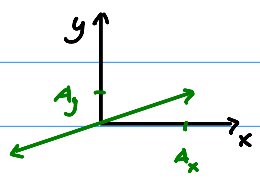
$$\ddot{x} = -\omega^2 x$$
$$\ddot{y} = -\omega^2 y$$

$$x(t) = A_x \cos(\omega t - \delta_x)$$
$$y(t) = A_y \cos(\omega t - \delta_y)$$

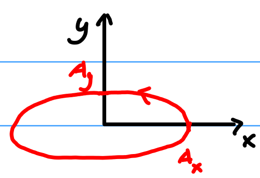
Redefinere t: $x(t) = A_x \cos(\omega t)$

$$y(t) = A_y \cos(\omega t - \delta)$$

$\delta = 0$:



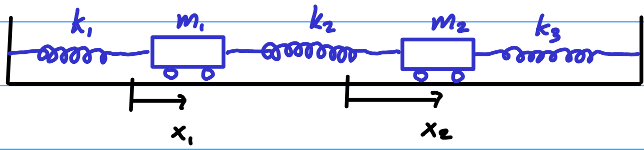
$\delta = \frac{\pi}{2}$:



Koblede oscillatorer (T. kap. 11)

Bil 1: $m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$

$= -(k_1+k_2)x_1 + k_2 x_2$



Bil 2: $m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - k_3 x_2$

$= k_2 x_1 - (k_2+k_3)x_2$

$$M\ddot{x} = -Kx, \qquad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

Ser etter løsninger på formen: $x(t) = \begin{bmatrix} \alpha_1 \cos(\omega t - \delta_1) \\ \alpha_2 \cos(\omega t - \delta_2) \end{bmatrix} = \begin{bmatrix} \alpha_1 \operatorname{Re} e^{i(\omega t - \delta_1)} \\ \alpha_2 \operatorname{Re} e^{i(\omega t - \delta_2)} \end{bmatrix}$

$$= \operatorname{Re} \underbrace{\underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{a} e^{i\omega t}}_{z(t)}, \qquad a_1 = \alpha_1 e^{-i\delta_1}, \quad a_2 = \alpha_2 e^{-i\delta_2}$$

Hvis $z(t)$ er en løsning av $M\ddot{z} = -Kz$, da er $x(t) = \operatorname{Re} z(t)$ en løsning av $M\ddot{x} = -Kx$.

$$M\ddot{z} = -Kz \;\Rightarrow\; M(i\omega)^2 a e^{i\omega t} = -Ka e^{i\omega t}$$
$$\Rightarrow\; (K - \omega^2 M)a = 0$$

Matriseteori: $Aa = 0$ og $a \neq 0 \Rightarrow \det A = 0$, fordi:

$$0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{matrix} A_{11}a_1 + A_{12}a_2 = 0 \\ A_{21}a_1 + A_{22}a_2 = 0 \end{matrix} \Rightarrow \begin{matrix} a_2 = -a_1 \frac{A_{11}}{A_{12}} \\ a_2 = -a_1 \frac{A_{21}}{A_{22}} \end{matrix}$$

$$\Rightarrow \frac{A_{11}}{A_{12}} = \frac{A_{21}}{A_{22}} \Rightarrow A_{11}A_{22} - A_{12}A_{21} = 0 \Rightarrow \det A = 0$$

Må ha $\det(K - \omega^2 M) = 0$.

Spesialtilfelle: $m_1 = m_2 = m$ og $k_1 = k_2 = k_3 = k$:

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$\det(K - \omega^2 M) = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = (2k - m\omega^2)^2 - k^2$$

$$= (k - m\omega^2)(3k - m\omega^2) = 0$$

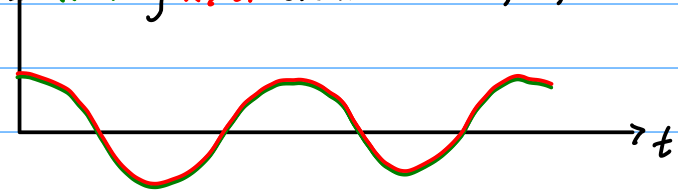
Løsninger: $\omega_1 = \sqrt{\frac{k}{m}}$, $\omega_2 = \sqrt{\frac{3k}{m}}$

Modus ω_1: $\omega_1 = \sqrt{\frac{k}{m}}$, $K - \omega_1^2 M = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$.

$$(K - \omega_1^2 M)a = 0 \Rightarrow k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_1 = a_2 \equiv A e^{-i\delta}$$

$$z(t) = \begin{bmatrix} A \\ A \end{bmatrix} e^{i(\omega_1 t - \delta)}, \quad x(t) = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_1 t - \delta), \quad \begin{aligned} x_1(t) &= A\cos(\omega_1 t - \delta) \\ x_2(t) &= A\cos(\omega_1 t - \delta) \end{aligned}$$

$x_1(t)$ og $x_2(t)$ oscillerer likt, i fase.

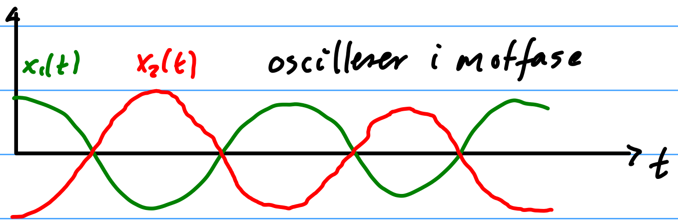


Modus ω_2: $\omega_2 = \sqrt{\frac{3k}{m}}$, $K - \omega_2 M = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}$, $(K - \omega_2 M)a = 0 \Rightarrow -k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$$a_1 = -a_2 \equiv A e^{i\delta}$$

$$z(t) = \begin{bmatrix} A \\ -A \end{bmatrix} e^{i(\omega_2 t - \delta)}, \quad x(t) = \begin{bmatrix} A \\ -A \end{bmatrix} \cos(\omega_2 t - \delta) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$x_1(t)$ $x_2(t)$ oscillerer i motfase



Generell løsning: $x(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2)$, $\omega_2 = \sqrt{3}\,\omega_1$

Spesialtilfelle svakt koblede oscillatorer:

$k_1 = k_3 = k, \quad k_2 << k. \qquad K = \begin{bmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{bmatrix}, \qquad K - \omega^2 M = \begin{bmatrix} k+k_2-m\omega^2 & -k_2 \\ -k_2 & k+k_2-m\omega^2 \end{bmatrix}$

$0 = \det(K - \omega^2 M) \Rightarrow \dots \Rightarrow \quad \omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{k+2k_2}{m}}$

Modus ω_1 : $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_1 = a_2 = C_1, \qquad z(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_1 t}$

Modus ω_2 : $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_1 = -a_2 = C_2, \qquad z(t) = C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega_2 t}$

Def $\omega_0 = \frac{\omega_1 + \omega_2}{2}, \quad \epsilon = \frac{\omega_2 - \omega_1}{2} \quad : \quad \begin{matrix} \omega_1 = \omega_0 - \epsilon \\ \omega_2 = \omega_0 + \epsilon \end{matrix}, \quad$ der $\epsilon << \omega_0$.

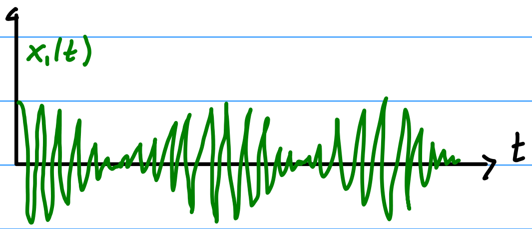
Generell løsn.: $z(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 - \epsilon)t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0+\epsilon)t}$

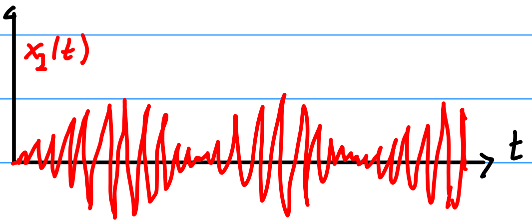
$$= \left(C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t} \right) e^{i\omega_0 t}$$

Anta $C_1 = C_2 = \frac{A}{2}$. $\quad z(t) = A \begin{bmatrix} \cos \epsilon t \\ -i \sin \epsilon t \end{bmatrix} e^{i\omega_0 t}, \qquad \begin{matrix} x_1(t) = A \cos\epsilon t \cos \omega_0 t \\ x_2(t) = A \cos \epsilon t \sin \omega_0 t \end{matrix}$

$\leftarrow \mathrm{Re}(-i e^{i\omega_0 t}) = \mathrm{Re}\, e^{i\omega_0 t - i\frac{\pi}{2}}$

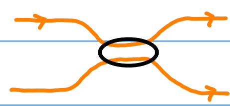
$= \cos(\omega_0 t - \frac{\pi}{2}) = \sin \omega_0 t$





Starter oscillasjon bil 1,
kobler svakt til bil 2,
kobler tilbake osv.

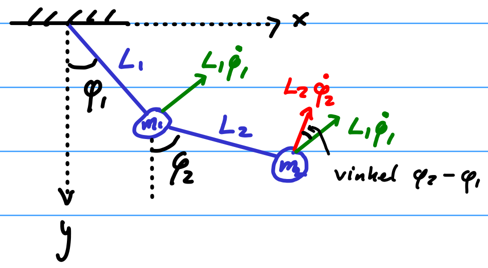
Eks:



piano

Eks: Dobbelpendel

Bruke φ_1 og φ_2 som koordinater.



$$U_1 = m_1 g L_1 (1-\cos\varphi_1)$$
$$U_2 = m_2 g [L_1(1-\cos\varphi_1) + L_2(1-\cos\varphi_2)]$$

$$T_1 = \tfrac{1}{2} m_1 (L_1\dot\varphi_1)^2, \qquad T_2 = \tfrac{1}{2} m_2 \left[(L_1\dot\varphi_1)^2 + (L_2\dot\varphi_2)^2 + 2L_1L_2\dot\varphi_1\dot\varphi_2\cos(\varphi_2-\varphi_1)\right]$$

(fordi $\vec v = \vec v_1 + \vec v_2 \Rightarrow \vec v^2 = \vec v_1^2 + \vec v_2^2 + 2\vec v_1\cdot\vec v_2$)

$\mathcal{L} = T - U = \dots$ gir Lagrange's ligninger...

Tilnærmelse: Små utslag $|\varphi_1|$ og $|\varphi_2|$, og små vinkelhast. $|\dot\varphi_1|$ og $|\dot\varphi_2|$.

$$\cos\varphi = 1 - \frac{\varphi^2}{2} + \dots$$

$$\Rightarrow T = \tfrac{1}{2}(m_1+m_2)L_1^2\dot\varphi_1^2 + \tfrac{1}{2}m_2L_2^2\dot\varphi_2^2 + m_2L_1L_2\dot\varphi_1\dot\varphi_2$$ ← her kan vi sette $\cos(\varphi_1-\varphi_2)\approx 1$ siden allerede multiplisert med små størrelser $\dot\varphi_1\dot\varphi_2$

$$U = \tfrac{1}{2}(m_1+m_2)gL_1\varphi_1^2 + \tfrac{1}{2}m_2gL_2\varphi_2^2$$ ← $\cos\varphi_{1,2} = 1 - \frac{\varphi_{1,2}^2}{2}$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi_1}\right) = \frac{\partial\mathcal{L}}{\partial\varphi_1} : \quad (m_1+m_2)L_1^2\ddot\varphi_1 + m_2L_1L_2\ddot\varphi_2 = -(m_1+m_2)gL_1\varphi_1$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi_2}\right) = \frac{\partial\mathcal{L}}{\partial\varphi_2} : \quad m_2L_1L_2\ddot\varphi_1 + m_2L_2^2\ddot\varphi_2 = -m_2gL_2\varphi_2$$

$$M\ddot\varphi = -K\varphi, \quad M = \begin{bmatrix}(m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2\end{bmatrix}, \quad K = \begin{bmatrix}(m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2\end{bmatrix}$$

Spesialtilfelle: $m_1 = m_2 = m$, $L_1 = L_2 = L$:

$$M = mL^2\begin{bmatrix}2 & 1 \\ 1 & 1\end{bmatrix}, \quad K = mL^2\begin{bmatrix}2\omega_0^2 & 0 \\ 0 & \omega_0^2\end{bmatrix}, \quad \omega_0 = \sqrt{\frac{g}{L}}$$

Kompleks representasjon: $\varphi(t) = \mathrm{Re}\, z(t)$, $\quad z(t) = a\, e^{i\omega t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t}$,
der ω er en ukjent "egenfrekvens".

$$K - \omega^2 M = mL^2 \begin{bmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{bmatrix}, \qquad (K - \omega^2 M)\, a = 0$$

$$\det(K - \omega^2 M) = 0 \;\Rightarrow\; 2(\omega_0^2 - \omega^2)^2 - \omega^4 = 0 \;\Rightarrow\; \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$
$$\Rightarrow\; \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2$$

Løsning $\omega_1^2 = (2 - \sqrt{2})\omega_0^2$:
$$K - \omega_1^2 M = mL^2 \begin{bmatrix} 2(1-(2-\sqrt{2}))\omega_0^2 & -(2-\sqrt{2})\omega_0^2 \\ -(2-\sqrt{2})\omega_0^2 & (1-(2-\sqrt{2}))\omega_0^2 \end{bmatrix}$$
$$= mL^2\omega_0^2(\sqrt{2}-1) \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \;\Rightarrow\; a_2 = \sqrt{2}\, a_1, \quad a_1 = A_1 e^{-i\delta_1}$$
$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \mathrm{Re}\left\{ \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} A_1 e^{-i\delta_1} e^{i\omega_1 t} \right\}$$
$$= \underline{A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_1 t - \delta_1)}$$

Løsning $\omega_2^2 = (2 + \sqrt{2})\omega_0^2$: $\quad \varphi(t) = \underline{A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_2 t - \delta_2)}$

Generell løsning, gyldig for små utslag og hastigheter:
$$\varphi(t) = A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_2 t - \delta_2)$$