Hamilton-mekanikk

Ligner på Lagrange-mek., men tar utg. punkt i Hamiltonfunksjonen $\mathcal{H}$ isf. $\mathcal{L}$. Vanligvis er $\mathcal{H}$ = totalenergien.

$$\mathcal{L} = T - U = \mathcal{L}(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n), \quad \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad i = 1, \dots, n.$$

Definer generalisert driv = kanonisk driv = driv konjugert til q_i : $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

Def. Hamiltonfunksjonen : $\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$

Uttrykker $\mathcal{H} = \mathcal{H}(q_1 \dots q_n, p_1 \dots p_n)$.

Dvs : $\mathcal{L} = \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$
$\mathcal{H} = \mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n)$

Rommet av $q_1, \dots, q_n, p_1, \dots, p_n$ kalles <u>faserommet</u>.

Generelt: $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \sum_{jk} \frac{1}{2} A_{jk} \dot{q}_j \dot{q}_k, \quad A_{jk} = A_{jk}(q_1 \dots q_n)$

for naturlige koordinater, dvs koordinater slik at sammenhengen $\vec{r} = \vec{r}(q_1, q_2, \dots)$ ikke har eksplisitt tidsavhengighet.

<u>1 koordinat</u> (konservativt) $\quad T(q, \dot{q}) = \frac{1}{2} A(q) \dot{q}^2$

$$\mathcal{L} = \frac{1}{2} A(q)\dot{q}^2 - U(q), \quad \text{gen. driv } p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = A(q)\dot{q}.$$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = A(q)\dot{q}^2 - \frac{1}{2}A(q)\dot{q}^2 + U(q) = T + U = \text{tot. energi}$$

Eliminerer $\dot{q}$ vha. $\dot{q} = \frac{p}{A(q)} = \dot{q}(q, p)$, $\quad \mathcal{H}(p, q) = p\dot{q}(q,p) - \mathcal{L}(q, \dot{q}(p,q))$

Bevegelsesligninger :
$$\frac{\partial \mathcal{H}}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \overbrace{\left[\frac{\partial \mathcal{L}}{\partial q} + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p} \frac{\partial \dot{q}}{\partial q}\right]}^{\text{fra kjerneregelen}} = -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p}$$

$$\boxed{\frac{\partial \mathcal{H}}{\partial q} = -\dot{p}}$$

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{q}(q,p) + p\frac{\partial \dot{q}}{\partial p} - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial p}$$

$$\boxed{\frac{\partial \mathcal{H}}{\partial p} = \dot{q}}$$ Hamiltons ligninger

<u>n koordinater</u>

Samme forutsetninger som vi brukte for Lagrangemek.: • holonome føringer
• ikke-føringskrefter på formen $\vec{F}_i = -\nabla_i U(t)$

$$q_i\ ,\ p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\mathcal{H} = \sum_i p_i \dot{q}_i - L \underset{\uparrow}{=} \mathcal{H}(q_1 \dots q_n, p_1 \dots p_n)$$

eliminerer $\dot{q}_i$ vha. $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Gir (tilsvarende til 10): $$\boxed{\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{og} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}\ ,\quad i = 1, \dots, n}$$

Generelt: $\mathcal{H} = \mathcal{H}(q_1 \dots q_n, p_1 \dots p_n, t)$.

$$\frac{d\mathcal{H}}{dt} = \sum_{i=1}^{n}\left[\frac{\partial \mathcal{H}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{H}}{\partial p_i}\frac{dp_i}{dt}\right] + \frac{\partial \mathcal{H}}{\partial t} \underset{\uparrow}{=} \frac{\partial \mathcal{H}}{\partial t}$$

$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$

Dvs.: Hvis $\mathcal{H}$ ikke avhenger eksplisitt av tiden, er <u>$\mathcal{H}$ bevart</u>.
For <u>naturlige</u> koordinater q_i, dvs. koordinater slik at sammenhengen $\vec{r}_\alpha = \vec{r}_\alpha(q_1 \dots q_n)$ ikke har eksplisitt tidsavhengighet, er $\mathcal{H}$ = <u>totalenergi</u> til systemet. (Bevis: Se ukesoppg.sett 10, oppg. 4)

Eks: En masse m i tyngdefeltet. $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - mgx$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$$

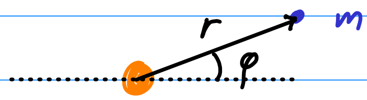
$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{m} - \frac{p^2}{2m} + mgx = \frac{p^2}{2m} + mgx$$

Mer generelt $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)$, gir $\mathcal{H} = \frac{p^2}{2m} + U(x)$

Hamiltons ligninger: $\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$, $\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{dU}{dx} = mg$

$$\Rightarrow \ddot{x} = \frac{\dot{p}}{m} = -\frac{1}{m}\frac{dU}{dx} \Rightarrow m\ddot{x} = -\frac{dU}{dx} = mg$$

Eks: Partikkel i sentralt kraftfelt:



$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \quad p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \quad (= r\, mr\dot{\varphi} = \ell = \text{spinn})$$

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\varphi} = \frac{p_\varphi}{mr^2}$$

$$\mathcal{H} = p_r\dot{r} + p_\varphi\dot{\varphi} - \mathcal{L} = m\dot{r}^2 + mr^2\dot{\varphi}^2 - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\varphi}^2 + U(r)$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + U(r)$$

$$= \frac{p_r^2}{2m} + \frac{1}{2}mr^2\frac{p_\varphi^2}{(mr^2)^2} + U(r) = \underline{\frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + U(r)}$$

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m}, \quad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = -\frac{dU}{dr} + \frac{p_\varphi^2}{mr^3}$$

$$m\ddot{r} = -\frac{dU}{dr} + \frac{p_\varphi^2}{mr^3} = -\frac{dU}{dr} + \frac{\ell^2}{mr^3} \quad \text{(som før!)}$$

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}, \quad \dot{p}_\varphi = -\frac{\partial \mathcal{H}}{\partial \varphi} = 0 \Rightarrow \ell = p_\varphi = \text{konst} \quad (\ell \text{ bevart})$$

Lagrange' lign. er 2. ordens diff.-ligninger.

Hamiltons lign. er 1. ordens diff-ligninger:

Def. $z = (q_1 \ldots q_n, p_1 \ldots p_n)$ ← punkt i faserommet

Hamiltons lign.: $\dot{z} = \mathcal{H}(z)$

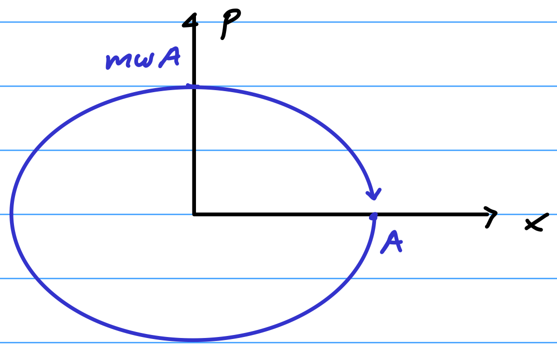
Eks: En harmonisk oscillator i faserommet $\mathcal{H} = T + U = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

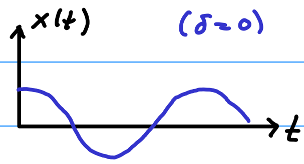
$\omega = \sqrt{\frac{k}{m}}$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x$$

$$\Rightarrow \ddot{x} = -\omega^2 x \Rightarrow x(t) = A\cos(\omega t - \delta) \quad \text{og} \quad p(t) = -m\omega A \sin(\omega t - \delta)$$

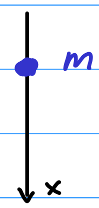
$$\frac{x^2}{A^2} + \frac{p^2}{(m\omega A)^2} = 1$$





Eks: m i fritt fall

$\mathcal{H} = \frac{p^2}{2m} - mgx$



Hamiltons lign. $\Rightarrow x = x_0 + \frac{p_0}{m} t + \frac{1}{2} g t^2, \quad p = p_0 + mgt$