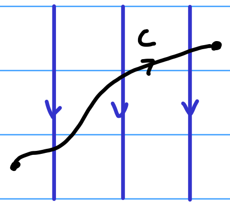
Vektoranalyse

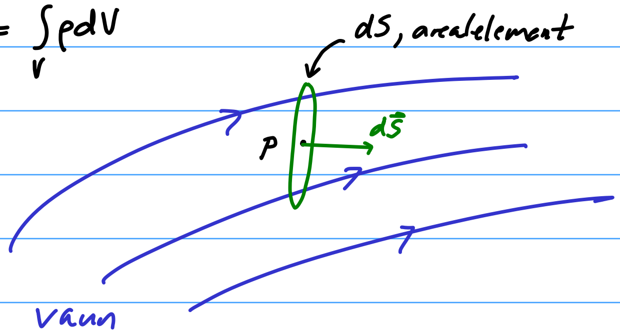
- Skalare funksjoner og vektorfelt
- Integraler 1D, 2D, 3D

Linjeintegral: F.eks. arbeid: $W = \int_C \vec{F} \cdot d\vec{l} = \int \vec{F} \cdot \underbrace{\vec{r}'(t)dt}_{d\vec{l}}$



Flakintegral: F.eks. strøm : $I = \int_S \vec{J} \cdot d\vec{S}$

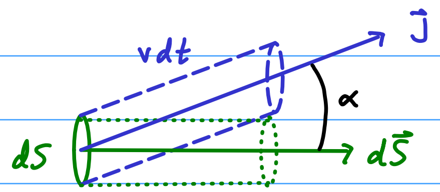
Volumintegral: F.eks masse : $m = \int_V \rho dV$



Def.: Strømtetthet: $\vec{J} = \rho \vec{v}$ ← hastighet
ρ: massetetthet

Enhet: $\frac{kg}{m^3} \cdot \frac{m}{s} = \frac{kg}{m^2 \cdot s}$

I punktet P er $\vec{J} \cdot d\vec{S}$ hvor mye vann (i kg) som strømmer gjennom dS per tidsenhet:



I løpet av dt fyller vannet den skjeve sylinderen. Volum: $dS \cdot vdt \cdot \cos\alpha$.

Masse $\rho \, dS \, v \, dt \cos\alpha = \rho v \, dS \cos\alpha \cdot dt = \vec{J} \cdot d\vec{S} \, dt$.

Dvs.: Per tidsenhet strømmer det $\vec{J} \cdot d\vec{S}$ masse vann gjennom dS.

$$I = \int_S \vec{J} \cdot d\vec{S} = \text{fluksen av } \vec{J} \text{ gjennom } S$$

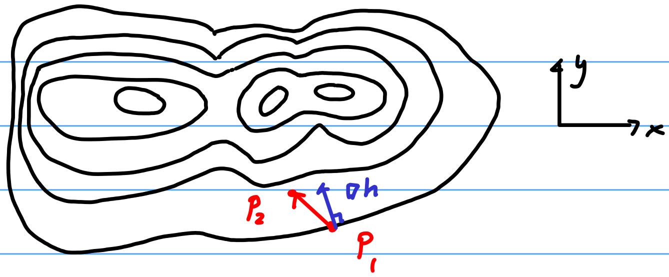
$$= \text{vannmengde gjennom } S \text{ per tidsenhet} = \text{strøm}$$

Gradient

$P_1 = (x, y)$

$P_2 = (x+dx, y+dy)$

Høyden $h(x,y)$ angitt med høydekurver



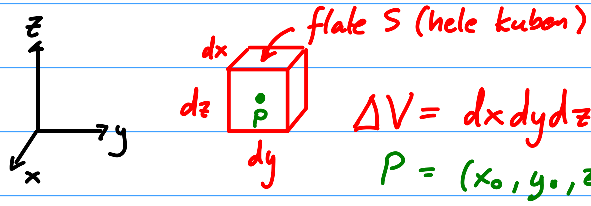
$$dh = \underbrace{\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy}_{\text{kjerneregelen}} = \underbrace{\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)}_{\nabla h} \cdot \underbrace{(dx, dy)}_{d\vec{l}} = \nabla h \cdot d\vec{l} = |\nabla h||d\vec{l}|\cos\theta$$

Høyden endrer seg raskest hvis vi går i retningen til ∇h. Da er $\theta = 0$, og stigningen $|\nabla h|$ hvis man går 1 i horisontal retning.

- 3D: $\nabla U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z} = \underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}_{\nabla} U$
- $\overbrace{\nabla U}^{\text{vektor}}$ (U: skalar)
- $\nabla U \perp$ flatene $U =$ konst.
- ∇U kan regnes ut i sylinder og sfæriske koord. Se formelsamling.

Divergens

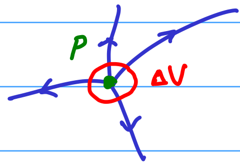
$$\operatorname{div}\vec{A} \overset{\text{def}}{=} \lim_{\Delta V \to 0} \frac{\oint_S \vec{A}\cdot d\vec{S}}{\Delta V}$$



$\Delta V = dx\,dy\,dz$

$P = (x_0, y_0, z_0)$

div $\vec{A}$ i P måler hvor mye $\vec{A}$ strømmer ut fra P:



$$\oint_S \vec{A}\cdot d\vec{S} = \oint_S \vec{A}\cdot\hat{n}\,dS$$

normalvektor (enhetsv.) til S

$$= \int_{\text{foran}} A_x(\text{foran})\,dy\,dz - \int_{\text{bak}} A_x(\text{bak})\,dy\,dz$$

$$- \int_{\text{venstre}} A_y(\text{venstre})\,dx\,dz + \int_{\text{høyre}} A_y(\text{høyre})\,dx\,dz$$

$$+ \int_{\text{topp}} A_z(\text{topp})\,dx\,dy - \int_{\text{bunn}} A_z(\text{bunn})\,dx\,dy$$

$$\to \frac{\partial A_x}{\partial x}dx\,dy\,dz + \frac{\partial A_y}{\partial y}dx\,dy\,dz + \frac{\partial A_z}{\partial z}dx\,dy\,dz$$

fordi $A_x(\text{foran}) - A_x(\text{bak}) = A_x(x_0 + dx/2, y_0, z_0) - A_x(x_0 - dx/2, y_0, z_0)$

$= \frac{\partial A_x}{\partial x}dx$ osv.

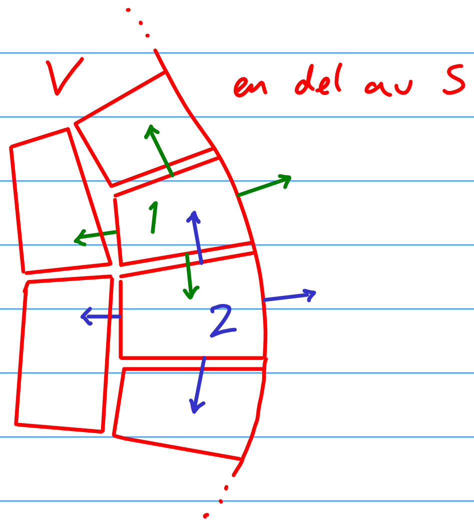
$$\Rightarrow \boxed{\operatorname{div}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial z} = \nabla\cdot\vec{A}}$$

- Fins også i syl. og sfær. koord. Slå opp!
- $\nabla\cdot\vec{A}$ er en skalar!
- $\nabla\cdot(a\vec{A} + b\vec{B}) = a\,\nabla\cdot\vec{A} + b\,\nabla\cdot\vec{B}$

Divergensteoremet

$$\boxed{\int_S \vec{A}\cdot d\vec{S} = \int_V \nabla\cdot\vec{A}\, dV}$$

for enhver lukket flate S,
S omslutter V



For 1: $\nabla\cdot\vec{A}\, dV_1 = \oint_{S_1} \vec{A}\cdot d\vec{S}$ (def. av divergens)

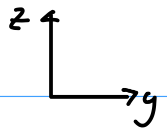
2: $\nabla\cdot\vec{A}\, dV_2 = \int_{S_2} \vec{A}\cdot d\vec{S}$

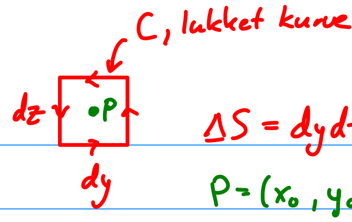
Gir $\int_V \nabla\cdot\vec{A}\, dV = \sum_i \oint_{S_i} \vec{A}\cdot d\vec{S}$

$= \oint_S \vec{A}\cdot d\vec{S}$

kansellering, se figur

Curl (virvling)





$\Delta S = dy\,dz$

$P = (x_0, y_0, z_0)$

$(\text{curl}\,\vec{A})_x \overset{\text{def}}{=} \lim\limits_{\Delta S \to 0} \dfrac{\oint_C \vec{A}\cdot d\vec{l}}{\Delta S}$: Måler virvlingen av $\vec{A}$ rundt P normalt på x-aksen.

$$\oint_C \vec{A}\cdot d\vec{l} = \int_{\text{nede}} A_y(\text{nede})\,dy - \int_{\text{oppe}} A_y(\text{oppe})\,dy - \int_{\text{venstre}} A_y(\text{venstre})\,dz + \int_{\text{høyre}} A_y(\text{høyre})\,dz$$

$$\rightarrow \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy\,dz$$

fordi $A_z(\text{høyre}) - A_z(\text{venstre}) = A_z(x_0, y_0 + dy/2, z_0) - A_z(x_0, y_0 - dy/2, z_0)$

$= \dfrac{\partial A_z}{\partial y}\,dy$ osv.

$$\Rightarrow (\text{curl}\,\vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \nabla \times \vec{A}.$$

- Fins i syl. og sfær. koord. Slå opp!
- $\nabla \times \vec{A}$ er en vektor
- $\nabla \times (a\vec{A} + b\vec{B}) = a\,\nabla\times\vec{A} + b\,\nabla\times\vec{B}$
- Merk at $\nabla \times (\nabla U) \equiv 0$ og $\nabla \cdot (\nabla \times \vec{A}) = 0$ (vis det!)

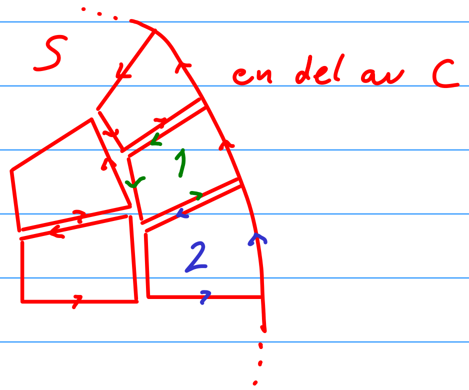
Stokes' teorem

$$\boxed{\oint_C \vec{A}\cdot d\vec{l} = \int_S (\nabla\times\vec{A})\cdot d\vec{S}}$$

for en hver lukket kurve C

C omslutter S

$$(\nabla\times\vec{A})\cdot d\vec{S}_1 = \oint_{C_1} \vec{A}\cdot d\vec{l}$$ (def. av curl)



$$(\nabla\times A)\cdot d\vec{S}_2 = \int_{C_2} \vec{A}\cdot d\vec{l}$$

$$\int(\nabla\times\vec{A})\cdot d\vec{S} = \sum_i \int_{C_i} \vec{A}\cdot d\vec{l}$$

$$= \oint_C \vec{A}\cdot d\vec{l}$$

intern kansellering (se figur)

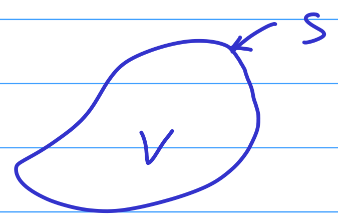
Laplace-operatoren

$$\nabla^2 U \equiv \nabla\cdot\nabla U = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U$$

Fins også i syl. og sfær. koord. Slå opp!

Eks: Kontinuitetsligningen

Antar at massen totalt sett er bevart.
Dvs.: En strøm av masse ut av S
må gå på bekostning av massen m i V.



$$\underbrace{\oint_C \bar{J}\cdot d\bar{S}}_{\text{strøm av } m \text{ ut av } S} = \underbrace{-\frac{dm}{dt}}_{\text{minking av massen } m \text{ i } V.}, \quad \bar{J} = \rho\bar{v} = \text{strømtetthet}$$

Vi har $m = \int_V \rho\, dV$, så $\oint_S \bar{J}\cdot d\bar{S} = -\frac{d}{dt}\int_V \rho\, dV$

$$= \int_V \left(-\frac{\partial\rho}{\partial t}\right) dV$$

↑ hvis V er fast (tidsuavh.)

Div. teoremet: $\oint_S \bar{J}\cdot d\bar{S} = \int_V \nabla\cdot\bar{J}\, dV$.

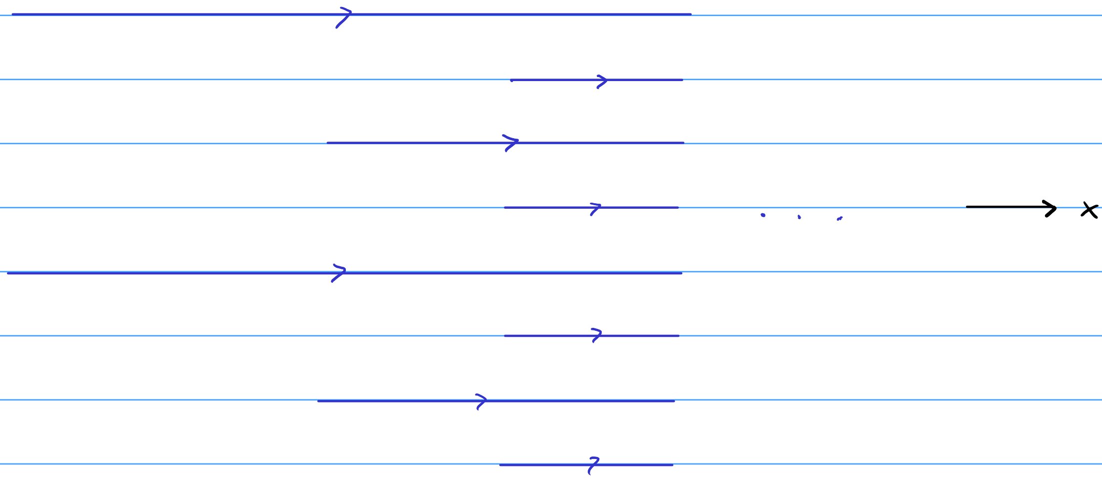
$$\Rightarrow \int_V \left(\nabla\cdot\bar{J} + \frac{\partial\rho}{\partial t}\right) dV = 0$$

Gyldig for vilkårlige V. Velger V mindre og mindre...

$$\Rightarrow \boxed{\nabla\cdot\bar{J} + \frac{\partial\rho}{\partial t} = 0}$$

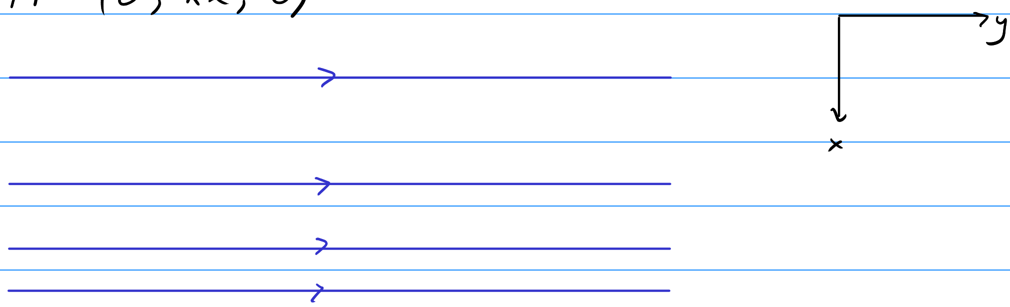
Kontinuitetsligningen
Lokal bevaringslov for masse

Eks: $\vec{A} = (kx, 0, 0)$, $k > 0$. Hvordan ser feltlinjene ut?
Finn $\nabla \cdot \vec{A}$ og $\nabla \times \vec{A}$ og vurder om svaret er rimelig.



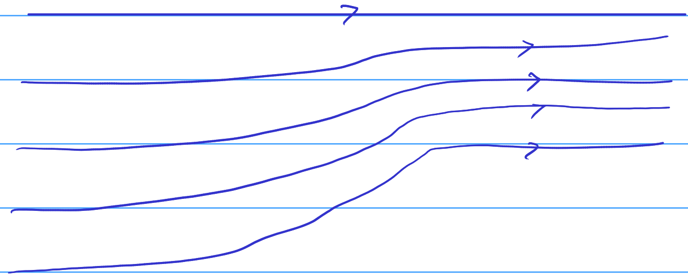
$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} = k \,, \qquad \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 0 & 0 \end{vmatrix} = 0$$

Eks: $\vec{A} = (0, kx, 0)$



$$\nabla \cdot \vec{A} = 0 \,, \qquad \nabla \times \vec{A} = k\hat{z}$$

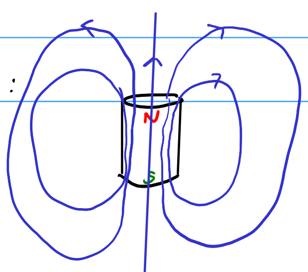
Eks:



$\nabla \cdot \vec{A} = 0$

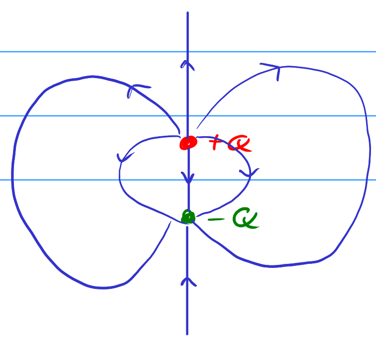
$\nabla \times \vec{A} \neq 0$

Eks:



$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{B} \neq 0$



$\nabla \cdot \vec{E} \neq 0$

$\nabla \times \vec{E} = 0$