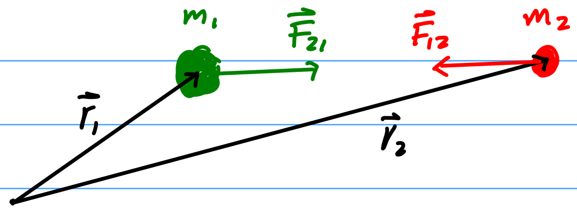
To-legeme-problemer, sentrale krefter (T. kap. 8)

F.eks.: månen og jorda, jorda og sola, elektron og proton (H-atom), to-atomig molekyl, ... (tilnærmede tolegeme-problemer)

To legemer:



Antar $\vec{F}_{12}$ og $\vec{F}_{21} = -\vec{F}_{12}$ konservative og sentrale.
rettet langs $\vec{r}_2 - \vec{r}_1$

Ser på m_1 og m_2 som punktlegemer, dvs størrelse $\ll |\vec{r}_2 - \vec{r}_1|$.

Pot. energi: $U(\vec{r}_1, \vec{r}_2)$. F.eks.: $U(\vec{r}_1, \vec{r}_2) = -\dfrac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|}$ for gravitasjon.

Anta at $U(\vec{r}_1, \vec{r}_2)$ bare avhenger av $|\vec{r}_1 - \vec{r}_2|$.
Def. $\vec{r} = \vec{r}_1 - \vec{r}_2$ = relativ posisjon. Kan da skrive $U = U(r)$.

$$\Rightarrow L = T - U = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)$$

Hvilke generaliserte koordinater bør vi bruke?
Svar: Massesenter $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$, $M = m_1 + m_2$

og rel. posisjon $\vec{r} = \vec{r}_1 - \vec{r}_2$. $\Big\}$ (*)

Gir $\vec{r}_1 = \vec{R} + \dfrac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \dfrac{m_1}{M}\vec{r}$. [Sjekk ved å sette inn i (*)]

$$T = \tfrac{1}{2} m_1 \left(\dot{\vec{R}} + \tfrac{m_2}{M}\dot{\vec{r}}\right)^2 + \tfrac{1}{2} m_2 \left(\dot{\vec{R}} - \tfrac{m_1}{M}\dot{\vec{r}}\right)^2 = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2}\frac{m_1m_2}{M}\dot{\vec{r}}^{\,2}$$

Def. redusert masse $\mu = \dfrac{m_1m_2}{M} = \dfrac{m_1m_2}{m_1+m_2}$

$T = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^{\,2}$.

$$\boxed{\text{Så } L = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^{\,2} - U(r)}$$

Generelt: For N partikler

Totalt driv: $\vec{P} = \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$, siden $\vec{R} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M}$,

dvs totalt driv er som om M befinner seg i massesenteret.

Newtons 2. lov: $\dot{\vec{P}} = \sum_\alpha \dot{\vec{p}}_\alpha = \sum_\alpha \sum_{\beta\neq\alpha} \vec{F}_{\alpha\beta} + \sum_\alpha \vec{F}_\alpha^{ext}$ ekstern kraft på part. α

kraft på part. α fra part. β

$$= \sum_\alpha \vec{F}_\alpha^{ext} \equiv \vec{F}^{ext}.$$

fordi $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$

Så Newtons 2 lov gjelder for tot. driv og ekstern kraft: $\vec{F}^{ext} = \dot{\vec{P}}$

Dette gir: $\vec{F}^{ext} = \dot{\vec{P}} = M\ddot{\vec{R}}$.

Fra før: $\vec{\tau}^{ext} = \dot{\vec{L}}$

Hvis ingen eksterne krefter:

$\dot{\vec{R}} =$ konst og $\vec{L} =$ konst.

Dvs.: Massesenteret beveger seg med konst. hastighet.

Vi velger derfor vanligvis et referansesystem der $\vec{R} = 0$,

så massesenteret er i origo.

Problem: $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$, $\vec{r} = (x, y, z)$, $r^2 = x^2 + y^2 + z^2$

$\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$, $\dot{\vec{r}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

Vi har redusert to-legeme-problemet til et ett-legeme-problem!

Lagrange: $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow -\frac{\partial U}{\partial x} = \frac{d}{dt}\mu\dot{x} = \mu\ddot{x}$, tilsvarende for y og z.

$$\boxed{-\nabla U(r) = \mu\ddot{\vec{r}}}$$

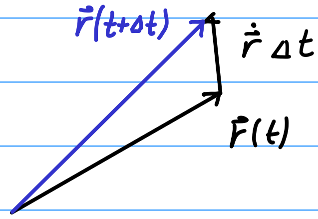
Spinn: $\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$, $\vec{r}_1 = \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \frac{m_1}{M}\vec{r}$

$$\vec{L} = \frac{m_1 m_2}{M^2}\left(m_2\, \vec{r} \times \dot{\vec{r}} + m_1\, \vec{r} \times \dot{\vec{r}}\right)$$

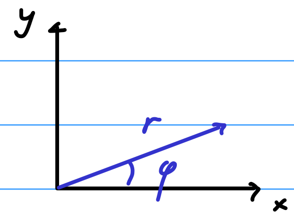
$= \mu\, \vec{r} \times \dot{\vec{r}} = \vec{r} \times \mu\dot{\vec{r}}$ = spinn for en partikkel med posisjon $\vec{r}$ og masse μ.

$\uparrow$ $\mu = \frac{m_1 m_2}{M}$ og $m_1 + m_2 = M$

Spinnet $\vec{L}$ er bevart, så $\vec{r} \times \dot{\vec{r}}$ = konst., og $\vec{r} \times \dot{\vec{r}}$ må da alltid være i samme retning. Dvs. banen $\vec{r}(t)$ er i ett plan, velger koordinater s.a. det er xy-planet.



Ett legeme, 2D: $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$



φ: $\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \Rightarrow 0 = \frac{d}{dt}(\mu r^2 \dot{\varphi})$

$\Rightarrow \mu r^2 \dot{\varphi}$ = konst $= \ell$, spinn bevart

$$\dot{\varphi} = \frac{\ell}{\mu r^2}$$

r: $\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \Rightarrow \mu r \dot{\varphi}^2 - \frac{dU}{dr} = \frac{d}{dt}\mu\dot{r} = \mu\ddot{r}$

$$\mu\ddot{r} = -\frac{dU}{dr} + \underbrace{\mu r \dot{\varphi}^2} = -\frac{dU}{dr} + \frac{\ell^2}{\mu r^3}$$

$\mu\frac{v_\varphi^2}{r}$, der $v_\varphi = r\dot{\varphi}$, så fiktiv "sentrifugalkraft".

Kan skrive dette som: $\mu \ddot{r} = -\frac{dV_{eff}}{dr}$,

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$
$$= U(r) + \frac{1}{2}\mu(r\dot{\varphi})^2$$

← kin. energi pga fart $r\dot{\varphi}$

Bevegelsen i radiell retn. er som for en partikkel i pot. $U_{eff}(r)$.

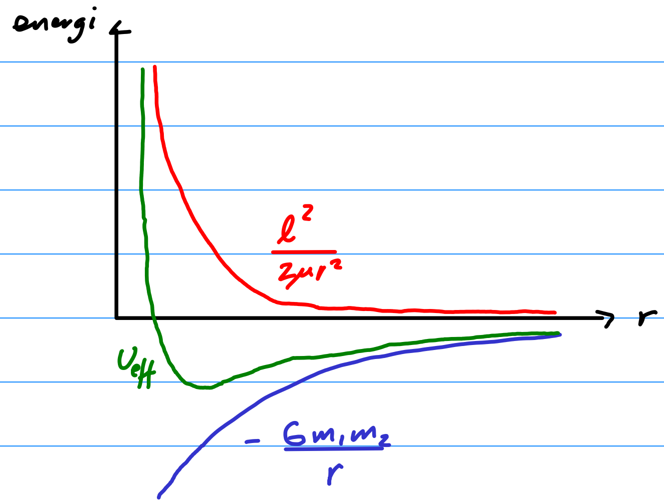
Tilfelle med gravitasjon, f.eks. komet rundt sola: Keplerproblemet.

$U(r) = -\frac{G m_1 m_2}{r}$:

$V_{eff}(r) = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$

r liten: $\frac{\ell^2}{2\mu r^2}$ dominerer

r stor: $\frac{-G m_1 m_2}{r}$ dominerer



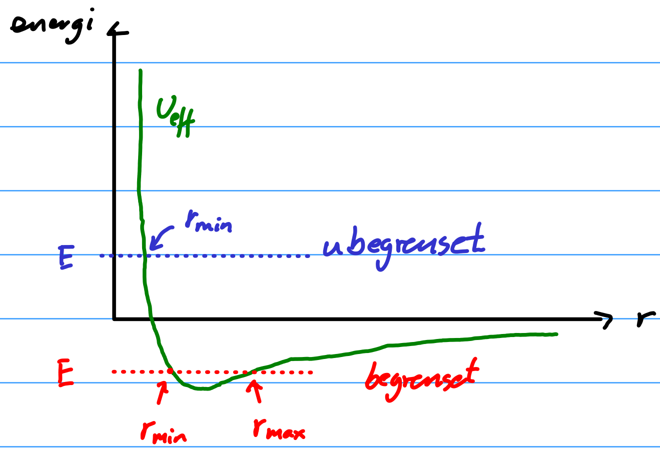
Bevaring av energi

$$\boxed{\frac{1}{2}\mu \dot{r}^2 + U_{eff}(r) = E \leftarrow \text{totalenergi}}$$

fordi $U_{eff}(r) = U(r) + \frac{1}{2}\mu(r\dot{\varphi})^2$

↑ kin. energi pga fart i φ-retn.

Har $\frac{1}{2}\mu \dot{r}^2 \geq 0$, så $U_{eff}(r) \leq E$



Utledning av baner

$$(*) \quad \mu \ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}, \qquad F(r) = -\frac{dU}{dr}$$

$$u = \frac{1}{r} \quad \Rightarrow \quad r = \frac{1}{u}$$

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \dot{\varphi}\frac{d}{d\varphi} = \frac{\ell}{\mu r^2}\frac{d}{d\varphi} = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}$$

$$\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\varphi}, \quad \ddot{r} = -\frac{\ell}{\mu}\frac{\ell u^2}{\mu}\frac{d^2u}{d\varphi^2} = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\varphi^2}$$

$$(*) \quad -\mu\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\varphi^2} = F + \frac{\ell^2 u^3}{\mu} \quad \Rightarrow \quad \underline{u''(\varphi) = -u(\varphi) - \frac{\mu F}{\ell^2 u^2(\varphi)}}$$

Gravitasjon: $F = -\frac{G m_1 m_2}{r^2} = -\frac{\gamma}{r^2}, \quad \gamma = G m_1 m_2$

$$F = -\gamma u^2$$

$$u''(\varphi) = -u(\varphi) + \frac{\mu\gamma}{\ell^2}$$

Def. $w(\varphi) = u(\varphi) - \frac{\mu\gamma}{\ell^2}$, gir $w''(\varphi) = -w(\varphi)$,

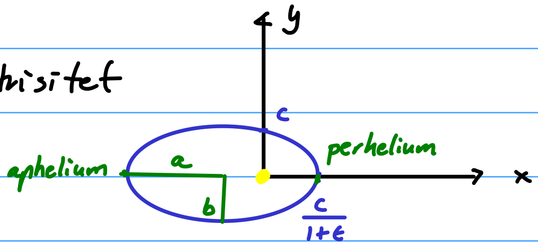
som har løsning $w(\varphi) = A\cos(\varphi - \cancel{\delta})$.

Løsning: $u(\varphi) = \frac{\mu\gamma}{\ell^2} + A\cos\varphi = \frac{\mu\gamma}{\ell^2}(1 + \epsilon\cos\varphi) = \frac{1}{r(\varphi)}$

$$\boxed{r(\varphi) = \frac{c}{1+\epsilon\cos\varphi}, \quad c = \frac{\ell^2}{\mu\gamma}, \quad \epsilon = \text{konst.}}$$

Bundne baner ($\epsilon < 1$)

ϵ = eksentrisitet



$$2a = \frac{c}{1+\epsilon} + \frac{c}{1-\epsilon} = \frac{2c}{1-\epsilon^2}$$

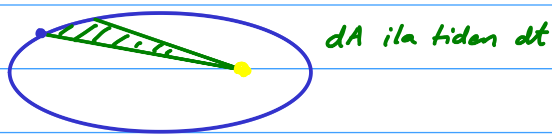
$$a = \frac{c}{1-\epsilon^2}$$

$$b = \frac{c}{\sqrt{1-\epsilon^2}}$$ (se ukesoppg)

Keplers lover:

1) Planetene følger ellipse-baner med sola i en av fokusene.

2) $\frac{dA}{dt} = \frac{\ell}{2\mu}$ = konst, der $\frac{dA}{dt}$ er raten som en linje fra planeten til sola sveiper over areal.



3) Periode $T^2 = \frac{4\pi^2}{GM_s} a^3$.

Fordi: $\frac{dA}{dt} = \frac{\ell}{2\mu}$, ellipseareal $A = \pi ab$, $T = \frac{A}{\frac{dA}{dt}} = \frac{\pi ab}{\frac{\ell}{2\mu}} = \frac{\pi ab\, 2\mu}{\ell}$

$$T^2 = \frac{4\pi^2 a^2 b^2 \mu^2}{\ell^2} = \frac{4\pi^2 a^4 \mu^2 (1-\epsilon^2)}{\ell^2} = \frac{4\pi^2 a^3 \mu^2 c}{\ell^2} = 4\pi^2 \frac{a^3 \mu}{\gamma}$$

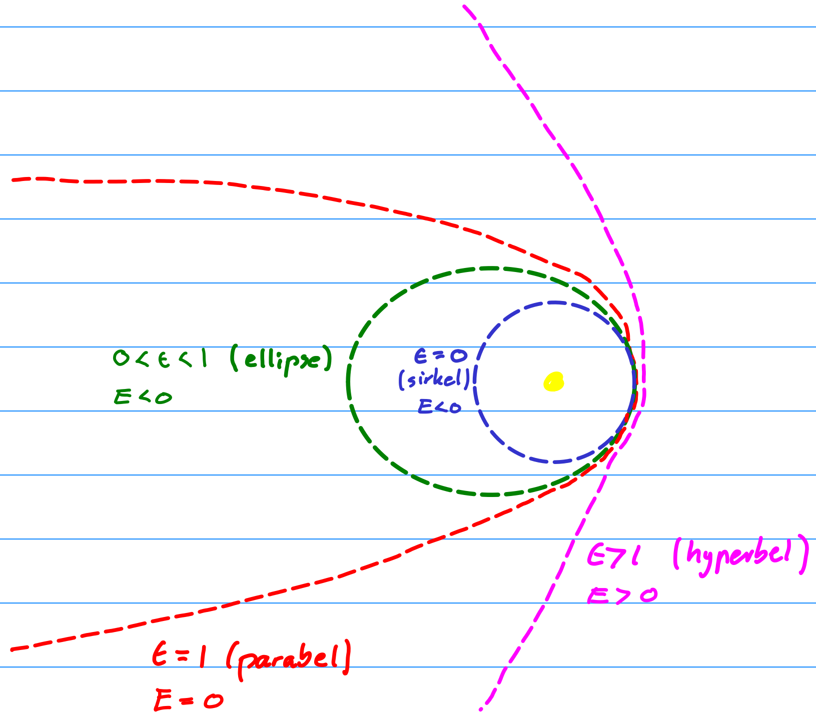
$\frac{b}{a} = \sqrt{1-\epsilon^2}$ $\qquad$ $c = \frac{\ell^2}{\mu\gamma}$

$$= \frac{4\pi^2}{GM_s} a^3$$

$\gamma = G M_s m_{planet} \approx G M_s \mu$

Ubundne baner $\epsilon \geq 1$

$$r(\varphi) = \frac{c}{1+\epsilon\cos\varphi} \qquad c = \frac{\ell^2}{G m_1 m_2 \mu}, \quad \mu = \text{red. masse}$$



For bundne og ubundne baner: $E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1)$, $\gamma = G m_1 m_2$,
fordi:

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{1}{2r_{min}}\left(-2\gamma + \frac{\ell^2}{\mu r_{min}}\right)$$

$$r_{min} = \frac{c}{1+\epsilon} = \frac{\ell^2}{\mu\gamma(1+\epsilon)}$$

$\uparrow\ c = \frac{\ell^2}{\mu\gamma}$

$$\Rightarrow E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1)$$