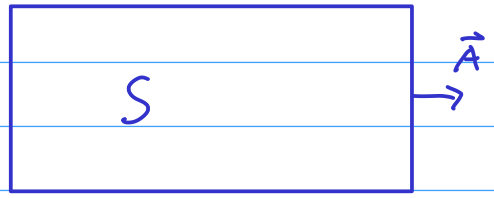
Mekanikk i akselererte systemer

Akselerasjon uten rotasjon:

S_0 (treghetssystem)



I S_0 gjelder Newtons lover: $m\ddot{\vec{r}}_0 = \vec{F}$

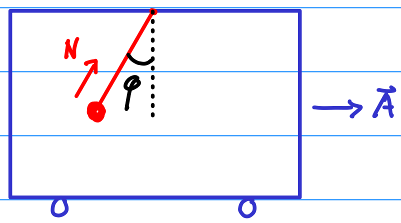
Sammenheng mellom $\vec{r}_0$ og $\vec{r}$: $\ddot{\vec{r}}_0 = \ddot{\vec{r}} + \vec{A} \Rightarrow \ddot{\vec{r}} = \ddot{\vec{r}}_0 - \vec{A}$

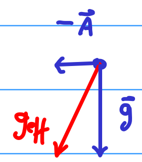
posisjon i S_0 posisjon i S

Dvs: $m\ddot{\vec{r}} = m\ddot{\vec{r}}_0 - m\vec{A} = \vec{F} - m\vec{A}$

$$\boxed{m\ddot{\vec{r}} = \vec{F} + \vec{F}_{inertial}, \quad \vec{F}_{inertial} = -m\vec{A}}$$

Eks: Pendel i akselererende bil

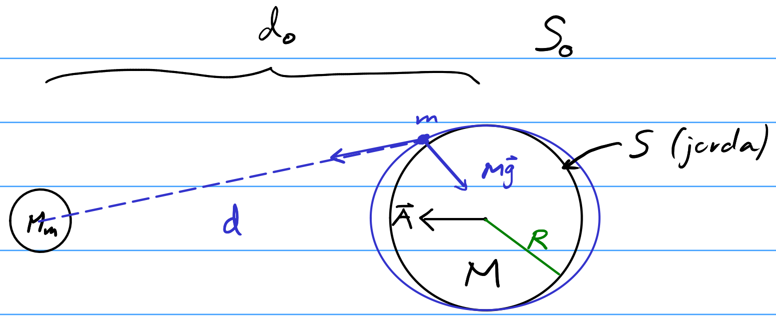




$$m\ddot{\vec{r}} = \vec{N} + m\vec{g} - m\vec{A} \Rightarrow m\ddot{\vec{r}} = \vec{N} + m(\vec{g} - \vec{A})$$
$$= \vec{N} + m\vec{g}_{eff}, \quad \vec{g}_{eff} = \vec{g} - \vec{A}$$

NB!!! Ikke bland sammen S_0 og S!
Vær tydelig på om du er i S_0 eller S!

Eks: Tidevann



$$\overset{\text{akselerasjon til jorda}}{}$$

S_0 : $M\vec{A} = -\dfrac{GMM_m}{d_0^2}\hat{d}_0 \Rightarrow \vec{A} = -\dfrac{GM_m}{d_0^2}\hat{d}_0$

S_0 : Kraft på m : $m\vec{g} - \dfrac{GM_m m}{d^2}\hat{d} + \vec{F}_{ng}$ ← andre krefter (fra vannet rundt)

$$S: \quad m\ddot{\vec{r}} = m\vec{g} - \frac{GM_m m}{d^2}\hat{d} + \vec{F}_{ng} + \overbrace{\frac{GM_m m}{d_0^2}\hat{d}_0}^{-m\vec{A} = \vec{F}_{inertial}}$$

$$\Rightarrow m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{tid} + \vec{F}_{ng}, \quad \vec{F}_{tid} = -GM_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right)$$

La m være nærmest månen. $\vec{F}_{tid} = -GM_m m\left(\dfrac{1}{(d_0-R)^2} - \dfrac{1}{d_0^2}\right)\hat{d}_0$ (rettet mot månen)

La m være lengst unna månen. $\vec{F}_{tid} = -GM_m m\left(\dfrac{1}{(d_0+R)^2} - \dfrac{1}{d_0^2}\right)\hat{d}_0$

$$= GM_m m\left(\frac{1}{d_0^2} - \frac{1}{(d_0+R)^2}\right)\hat{d}_0 \quad \text{(rettet fra månen)}$$

Disse er omtrent like : $\dfrac{1}{(d_0-R)^2} - \dfrac{1}{d_0^2} = \dfrac{1}{d_0^2}\left[\dfrac{1}{(1-\frac{R}{d_0})^2} - 1\right]$

$$\approx \frac{1}{d_0^2}\left(1 + 2\frac{R}{d_0} - 1\right) = \frac{2R}{d_0^3}$$

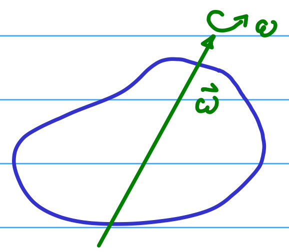
$(1+u)^p \approx 1 + pu$ når $u \ll 1$.

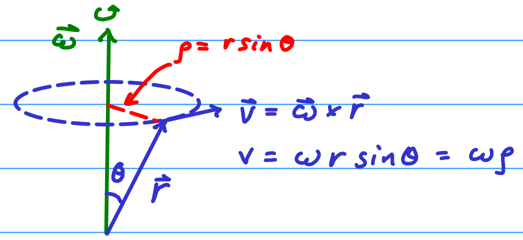
$$\frac{1}{d_0^2} - \frac{1}{(d_0+R)^2} \approx -\frac{2R}{d_0^3}$$

$$\underline{F_{tid} = \frac{2GM_m m R}{d_0^3}}$$

Rotasjon:

- Eulers teorem: Enhver bevegelse av et fast legeme slik at ett punkt er fiksert, er en rotasjon med akse gjennom punktet.





- Et punkt $\vec{r}$ på legemet beveger seg med hastighet $\vec{v} = \vec{\omega} \times \vec{r}$.
- En vektor fiksert i legemet har tidsderivert $\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}$ sett fra ref.systemet som er i ro.

Tidsderiverte i roterende systemer

S_0 : treghetssystem/inertialsystem : (x_0, y_0, z_0).

S : system som roterer med vinkelhast. $\vec{\Omega}$ relativt til S_0 : (x, y, z).

Antar S_0 og S har felles origo.

Ser på en vektor $\vec{Q}$. (Kan være f.eks. en posisjon, hastighet, kraft, ...)

$\left(\frac{d\vec{Q}}{dt}\right)_{S_0}$: tidsderivert sett fra S_0

$\left(\frac{d\vec{Q}}{dt}\right)_{S}$: tidsderivert sett fra S.

La $\vec{e}_1, \vec{e}_2, \vec{e}_3$ være tre ortogonale enhetsvektorer fiksert i S.

Skriver : $\vec{Q} = \sum_{i=1}^{3} Q_i \vec{e}_i$

Deriverer : $\left(\frac{d\vec{Q}}{dt}\right)_S = \sum_i \frac{dQ_i}{dt}\vec{e}_i$ ($\vec{e}_i$ er konstante i S)

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\vec{e}_i + \sum_i Q_i \underbrace{\left(\frac{d\vec{e}_i}{dt}\right)_{S_0}}_{\vec{\Omega}\times\vec{e}_i} = \sum_i \frac{dQ_i}{dt}\vec{e}_i + \vec{\Omega}\times\vec{Q}$$

$$\Rightarrow \boxed{\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_{S} + \vec{\Omega}\times\vec{Q}}$$

Newtons 2. lov i roterende system

I S_0 : $\quad m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F} \quad (*)$

$$\left(\frac{d\vec{\Omega}}{dt}\right)_{S_0} = \left(\frac{d\vec{\Omega}}{dt}\right)_S + \vec{\Omega}\times\vec{\Omega} = \left(\frac{d\vec{\Omega}}{dt}\right)_S$$

$\Rightarrow$ Hvis $\vec{\Omega}$ er konst. i S_0 er $\vec{\Omega}$ konst i S.

Har : $\quad \left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right] = \left(\frac{d}{dt}\right)_S\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right] + \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

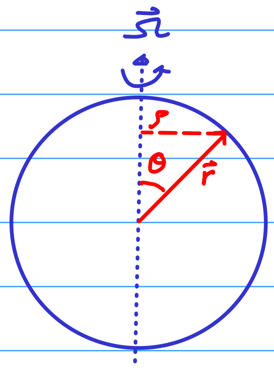
Definerer : $\quad \dot{\vec{Q}} \equiv \left(\frac{d\vec{Q}}{dt}\right)_S$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \ddot{\vec{r}} + \vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Newtons 2. lov $(*)$ gir i ref. syst. S :

$$\boxed{m\ddot{\vec{r}} = \vec{F} + \underbrace{2m\dot{\vec{r}}\times\vec{\Omega}}_{\vec{F}_{cor}} + \underbrace{m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}_{\vec{F}_{cf}}}$$

Sentrifugalkraft



$$F_{cf} = m\Omega r\sin\theta\cdot\Omega = m\Omega^2 r\sin\theta$$
$$= m\Omega^2\rho = m\frac{v_\varphi^2}{\rho}, \quad v_\varphi = \Omega\rho$$

$$\vec{F}_{cf} = m\Omega^2\rho\,\hat{\rho}$$

Eks: Akselerasjon i fritt fall: $\vec{g} \neq \vec{g}_0$ ← pga. gravitasjon

($\vec{g}$: pga gravitasjon + sentrifugalkraft)

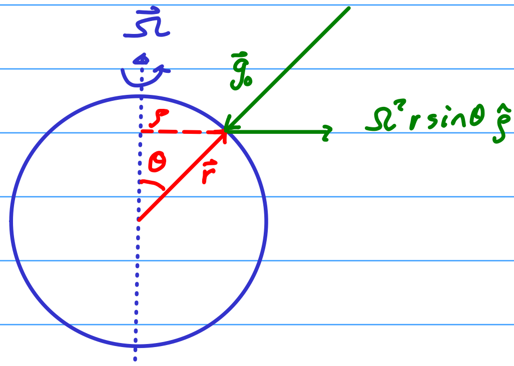
S = jorda

$m\ddot{\vec{r}} = \vec{F}_{grav} + \vec{F}_{cf}$ (står i ro på jorda, så $\dot{\vec{r}} = 0 \Rightarrow \vec{F}_{cor} = 0$)

$$\vec{F}_{grav} = -\frac{GMm\hat{r}}{r^2} = m\vec{g}_0 \ , \quad \vec{g}_0 = -\frac{GM\hat{r}}{r^2}$$

$$\vec{F}_{cf} = m\Omega^2 \rho \hat{\rho} = m\Omega^2 r \sin\theta \, \hat{\rho}$$

$$\vec{g} = \vec{g}_0 + \Omega^2 r \sin\theta \, \hat{\rho} \ :$$



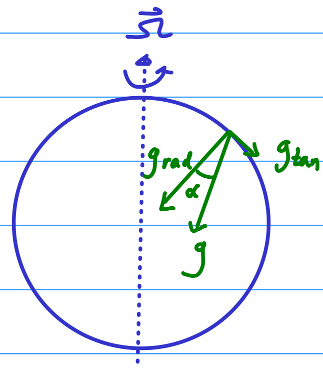
$$g_{rad} = g_0 - \Omega^2 r \sin^2\theta$$
$$g_{tang} = \Omega^2 r \sin\theta \cos\theta$$

For jorda: $\Omega^2 r = 0{,}034 \ \frac{m}{s^2}$

$$\tan\alpha = \frac{g_{tan}}{g_{rad}} \approx \frac{\Omega^2 r \sin\theta\cos\theta}{g_0}$$

$$\alpha \approx \frac{\Omega^2 r \sin 2\theta}{2 g_0}$$

$$\alpha_{max} \approx \frac{\Omega^2 r}{2 g_0} \approx 0{,}1^\circ$$



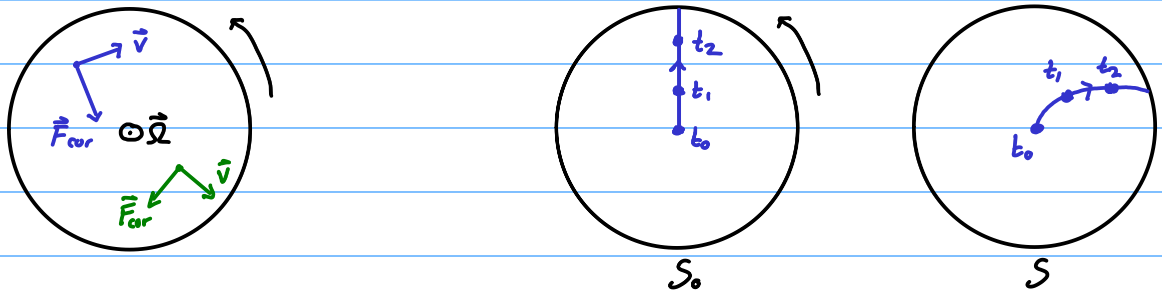
Kaller som regel retn. til $\vec{g}$ "vertikal", og retn. $\perp$ på "horisontal".

Coriolis-kraft

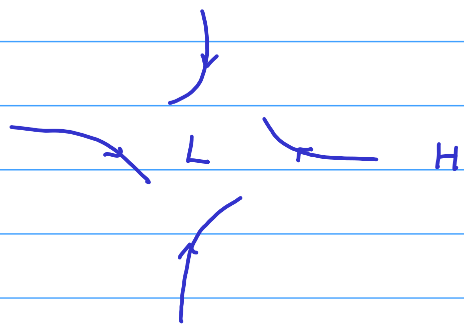
$$\vec{F}_{cor} = 2m\dot{\vec{r}} \times \vec{\Omega} = 2m\vec{v} \times \vec{\Omega}, \quad \vec{F}_{cor} \perp \vec{v}.$$

[Tilsvarende til: $\vec{F}_{magn} = q\vec{v} \times \vec{B}$]

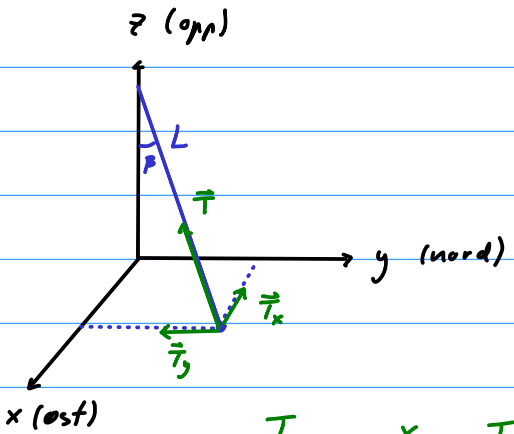
[Maks akselerasjon pga Coriolis: $a_{max} = 2v\Omega = 0{,}007\ m/s^2$
f.eks.: $v = 50\ \frac{m}{s}$, Ω jordrotasjon]



Nordlige halvkule: Vinder bøyer av mot høyre.



Foucault-pendel



$$m\ddot{\vec r} = \vec T + m\vec g + 2m\dot{\vec r}\times\vec\Omega$$

[inkluderer sentrifugalkraften $\vec F_{cf}$

$$\dot{\vec r}\times\vec\Omega = \begin{vmatrix}\hat x & \hat y & \hat z\\ \dot x & \dot y & \dot z\\ \Omega_x & \Omega_y & \Omega_z\end{vmatrix} = \hat x(\dot y\Omega_z - \dot z\Omega_y) + \hat y(\dot z\Omega_x - \dot x\Omega_z) + \hat z(\ \dots\ )$$

$$\approx \hat x\,\dot y\,\Omega_z - \hat y\,\dot x\,\Omega_z + \hat z(\dots)$$

$$= \hat x\,\dot y\,\Omega\cos\theta - \hat y\,\dot x\,\Omega\cos\theta$$

$\frac{T_x}{T} = -\frac{x}{L}$, $\frac{T_y}{T} = -\frac{y}{L}$

$T \approx mg$

(θ = vinkel ned fra nordpolen, som før)

$$m\ddot x = T_x + 2m\dot y\Omega_z \quad\Rightarrow\quad \ddot x = -g\frac{x}{L} + 2\dot y\Omega_z \quad\Rightarrow\quad \ddot x - 2\Omega_z\dot y + \omega_0^2 x = 0$$

$$m\ddot y = T_y - 2m\dot x\Omega_z \quad\Rightarrow\quad \ddot y = -g\frac{y}{L} - 2\dot x\Omega_z \quad\Rightarrow\quad \ddot y + 2\Omega_z\dot x + \omega_0^2 y = 0$$

[def.: $\omega_0 = \sqrt{\frac{g}{L}}$

Def.: $\eta = x + iy$. $\quad\ddot\eta + 2i\Omega_z\dot\eta + \omega_0^2\eta = 0$. Andre ordens diff.lign., homogen og med konst. koeffisienter.

Prøveløsn.: $\eta(t) = e^{-i\alpha t}$: $\left[(-i\alpha)^2 + 2i\Omega_z(-i\alpha) + \omega_0^2\right]e^{-i\alpha t} = 0$

$$\Rightarrow \alpha^2 - 2\Omega_z\alpha - \omega_0^2 = 0$$

$$\Rightarrow \alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2} \approx \Omega_z \pm \omega_0$$

[$\omega_0 >> \Omega_z$

Generell løsn.: $\eta(t) = e^{-i\Omega_z t}\left(C_1 e^{-i\omega_0 t} + C_2 e^{i\omega_0 t}\right)$

Initialbetingelse, f.eks: $\eta(0) = A + i0 = A$ (dvs. $x(0) = A$, $y(0) = 0$)

og pendel i ro ved $t = 0$: $\eta'(0) = 0$

$$\eta(0) = A \Rightarrow C_1 + C_2 = A$$

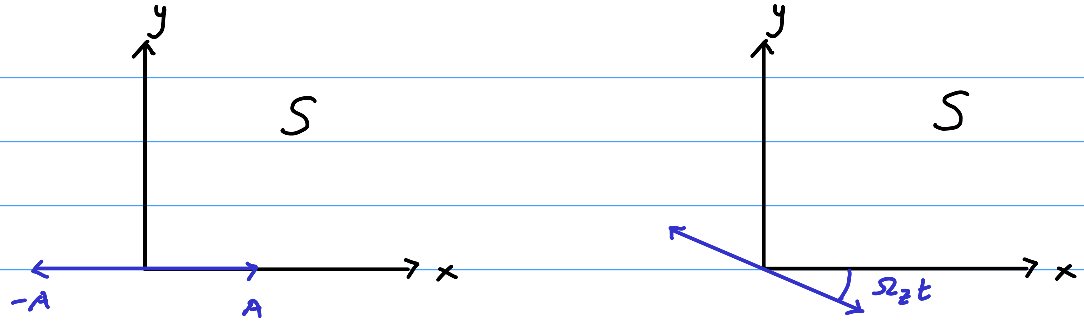
$$\eta'(0) = 0 \Rightarrow C_1(-i\Omega_z - i\omega_0) + C_2(-i\Omega_z + i\omega_0) = 0$$

$$A\Omega_z + (C_1 - C_2)\omega_0 = 0$$

$$C_1 - C_2 = \frac{A\Omega_z}{\omega_0} << A = C_1 + C_2$$

Dvs.: $C_1 - C_2 \approx 0$, $C_1 + C_2 = A \Rightarrow C_1 \approx C_2 = \frac{A}{2}$

Løsning: $\underline{\eta(t) = A e^{-i\Omega_z t}\cos\omega_0 t}$

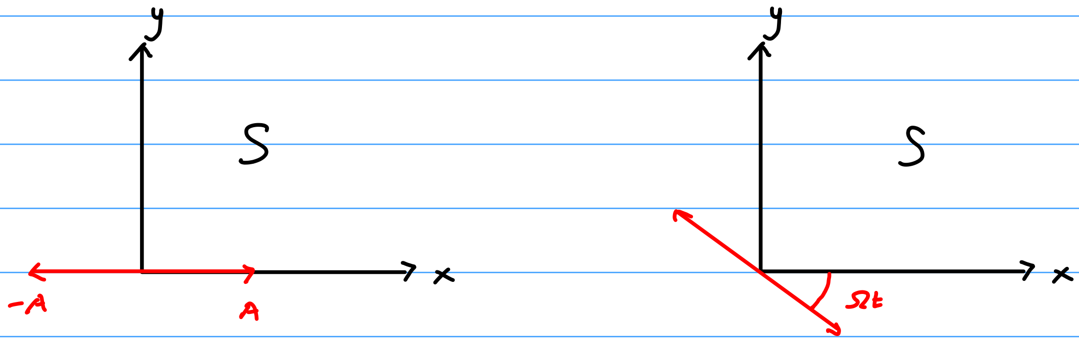


$t=0$ $\qquad$ $t>0$

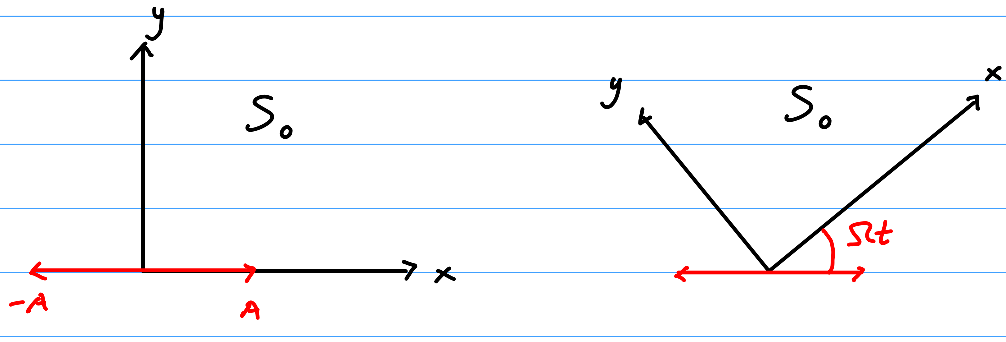
Oslo: ~60° nord, $\theta = 90° - 60° = 30°$. $\quad \Omega_z = \Omega \cdot \cos 30° = \frac{\sqrt{3}}{2}\Omega$

dvs $\frac{\sqrt{3}}{2} \cdot 90° \approx 78°$ på 6 timer.

Nordpolen:



$t=0$ $\qquad$ $t>0$



$t=0$ $\qquad$ $t>0$