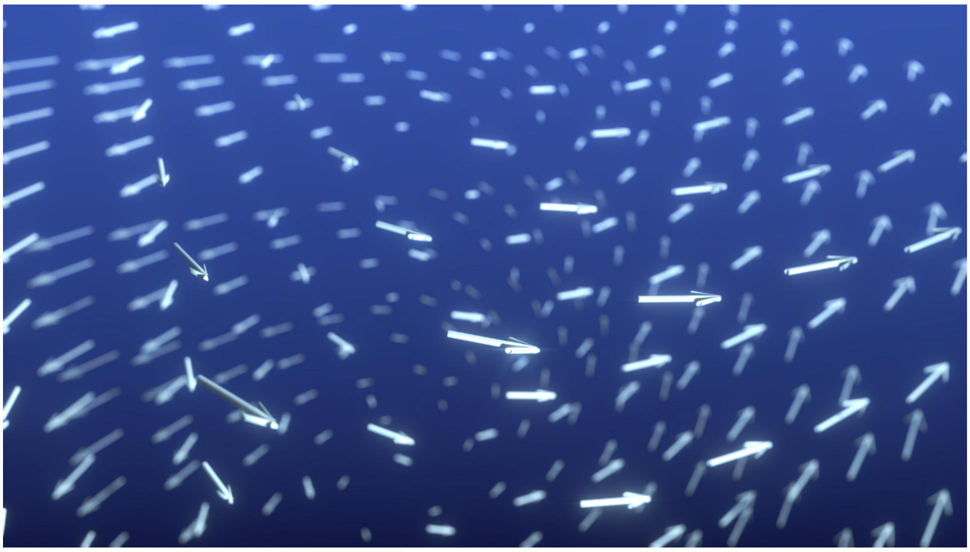
Week 1 – Mathematical warm up



One of the principal objects of research in my department of knowledge is to find the point of view from which the subject appears in the greatest simplicity.

Josiah Willard Gibbs[1]

Welcome to the first problem set in FYS1120 this year. We're starting out with some exercises on vector calculus and following up with the relevant theorems you need to know this semester. Before you get started, we suggest you have a look at the notes on vector calculus that we've put up on our web pages:

http://mindseye.no/fys1120/notes/

The first exercises are basic vector addition, subtraction and products. This might seem trivial to some of you, so you may skip ahead to exercise 1.2 if you'd like, but remember that some repetition never hurts.

Exercise 1.1: Basic vector calculus

Two vectors are defined as $\mathbf{v} = 1\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $u = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$. This is the equivalent to $\mathbf{v} = [1, 4, -2]$ and $\mathbf{u} = [-3, 2, 8]$, but in this course we're mostly using the first type of notation.[2]

a) Calculate $\mathbf{v} + 2\mathbf{u}$.

Answer: $\mathbf{v} + 2\mathbf{u} = \mathbf{v} = 1\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}} + 2(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}) = -5\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 14\hat{\mathbf{k}}$

b) Calculate $3\mathbf{u} - 2\mathbf{v}$

[1] The quote is from A L Mackay, Dictionary of Scientific Quotations (London, 1994), via Wikiquote

[2] If you'd like to know more about the notation used in this course, check out our page on notation: http://mindseye.no/fys1120/tips-and-tricks/

Answer: $3\mathbf{u} - 2\mathbf{v} = 3(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}) - 2(1\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = -11\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 28\hat{\mathbf{k}}$

c) Calculate $\mathbf{v} \cdot \mathbf{u}$.

Answer: $\mathbf{v} \cdot \mathbf{u} = 1 \cdot (-3) + 4 \cdot 2 - 2 \cdot 8 = -11$

d) Calculate $\mathbf{w} = \mathbf{v} \times \mathbf{u}$.

Answer: $\mathbf{v} \times \mathbf{u} = 36\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 14\hat{\mathbf{k}}$

e) In what direction does $\mathbf{w}$ point relative to the two vectors $\mathbf{v}$ and $\mathbf{u}$?

It is important that you understand how the different vector operators such as addition and multiplication works. While working with real problems, however, you will have to perform such calculations on millions of vectors. This is where the aid of a computer comes in.

Let us now do the calculations above quickly. First you'll need to have Python and several packages installed on your computer. Have a look at our guide, *Python for Electromagnetism*, before you start out with these exercises. This guide is available on our webpages:

http://mindseye.no/fys1120/notes/python/

f) Start up a terminal and run this command:

```
ipython --pylab
```

g) With this open, store the two vectors in memory:

```
v = array([1, 4, -2])
u = array([-3, 2, 8])
```

h) Calculate the same as you did on paper above, and check that your results match. You will need the following functions:

```
dot(u, v) # returns the dot product of two vectors
cross(u, v) # returns the cross product of two vectors
```

Exercise 1.2: Flux and Divergence

Divergence is a measure of the *expansion* of the vector field and it's denoted $\boldsymbol{\nabla} \cdot \mathbf{F}$. Let's first get some practice calculating the divergence. Find $\boldsymbol{\nabla} \cdot \mathbf{F}$ of the following fields.

a) $\mathbf{F} = x\hat{\mathbf{i}}$

Solution:

$$\boldsymbol{\nabla} \cdot x\hat{\mathbf{i}} = \hat{\mathbf{i}}\frac{\partial}{\partial x} \cdot x\hat{\mathbf{i}} = \frac{\partial}{\partial x}x = 1$$

Answer:

$$\nabla \cdot \mathbf{F} = 1$$

b) $\mathbf{F} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Solution:

$$\nabla \cdot \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

Answer:

$$\nabla \cdot \mathbf{F} = 3$$

c) $\mathbf{F} = \nabla \times \mathbf{G}$. That is, find the divergence of a field which is itself the curl of a field $\mathbf{G}$.

Hint: Start from the definition of the curl of $\mathbf{G}$.

Solution:

$$\begin{aligned} \mathbf{F} &= \nabla \times \mathbf{G} && (1) \\ &= \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right) && (2) \end{aligned}$$

such that

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{G}) && (3) \\ &= \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \cdot \left[\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)\right] && (4) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right) && (5) \\ &= \frac{\partial^2 G_z}{\partial x \partial y} - \frac{\partial^2 G_y}{\partial x \partial z} + \frac{\partial^2 G_x}{\partial y \partial z} - \frac{\partial^2 G_z}{\partial y \partial x} + \frac{\partial^2 G_y}{\partial z \partial x} - \frac{\partial^2 G_x}{\partial z \partial y} && (6) \\ &= 0 && (7) \end{aligned}$$

Answer:

$$\nabla \cdot \mathbf{F} = 0$$

where we have used the fact that the order of partial differentiation does not matter.

Check your results on the fields in (a) and (b) with your intuition[3]. You might want to plot or make a rough sketch of the fields. Two dimensions will do.

Solution:

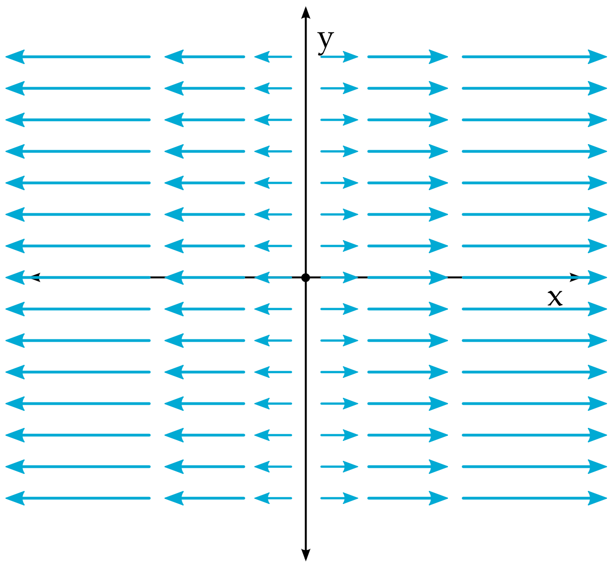


Figure 1: If we draw a little sphere in this field (a) (circle in 2D), it would have a positive flux.

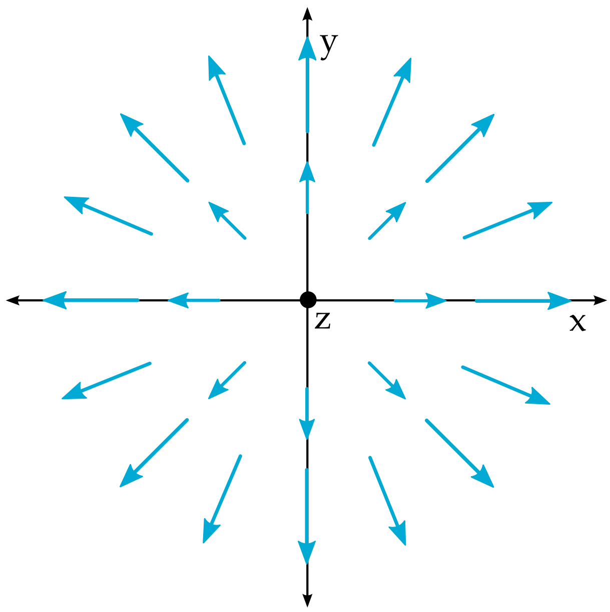


Figure 2: Field from (b). Conclusion: Same as in (a).

The vector field from a point charge or a sphere of charge centered at the origin have the form $\mathbf{F} = \hat{\mathbf{r}}/r^2$. In Cartesian coordinates $r^2 = x^2 + y^2 + z^2$ and

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}},$$

such that

$$\mathbf{F} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}.$$

d) Calculate the flux of $\mathbf{F}$ trough a sphere of radius r centered at the origin.

Solution:

$$\begin{aligned}
\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{A} &= \oint_{\mathcal{S}} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{A} && (8)\\
&= \oint_{\mathcal{S}} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{A} && (9)\\
&= \frac{1}{r^2} \oint_{\mathcal{S}} dA && (10)\\
&= \frac{4\pi r^2}{r^2} && (11)\\
&= 4\pi. && (12)
\end{aligned}$$

Using the fact that $\hat{\mathbf{r}}$ is parallel with $d\mathbf{A}$ along the sphere such that $\hat{\mathbf{r}} \cdot d\mathbf{A} = |\hat{\mathbf{r}}||d\mathbf{A}| = dA$ and that $\oint_{\mathcal{S}} dA$ is the surface area of $\mathcal{S}$.

Answer:

$$\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{A} = 4\pi$$

e) Show that the divergence of $\mathbf{F}$ is zero everywhere except at the origin.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{F} &= \frac{\partial}{\partial x} \frac{x}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial}{\partial y} \frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial}{\partial z} \frac{z}{(x^2 + y^2 + z^2)^{3/2}} && (13)\\
&= \frac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2) - 3y^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2) - 3z^2}{(x^2 + y^2 + z^2)^{5/2}} && (14)\\
&= \frac{3(x^2 + y^2 + z^2) - -3x^2 - 3y^2 - 3z^2}{(x^2 + y^2 + z^2)^{5/2}} && (15)\\
&= 0, && (16)
\end{aligned}$$

for $x, y, z \neq 0$.

f) Now use this result and the divergence theorem to calculate the flux trough any closed surface which does not enclose the origin.

Solution: The divergence theorem states

[3] Ask yourself weather a tiny sphere centered at the point in question would have a positive or negative flux.

$$\oint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{dA} = \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{F}dV, \tag{17}$$

but since

$$\boldsymbol{\nabla}\cdot\mathbf{F} = 0$$

as long as we're not enclosing the origin

$$\oint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{dA} = \int_{\mathcal{V}} 0dV = 0 \tag{18}$$

for all surfaces which are not enclosing the origin.

In 1783, Charles Augustin de Coulomb used a torsion balance[4] to study the behavior of electrical forces. This lead him to derive the famous formula for electrostatic forces between charges, known as Coulomb's law,

$$\mathbf{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}},$$

where $\mathbf{r}$ is the unit vector pointing from the charge q_1, placed at the origin, to the charge q_2 and ε_0 is the electrical permittivity in vacuum.[5].

Five years later, the British scientist Henry Cavendish performed a similar experiment where he measured the gravitational constant. This is used in a law very similar to Coulomb's, namely the universal law of gravitation,

$$\mathbf{F}_g = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}\,.$$

Even though the laws are very similar, they appear quite differently in our daily lives. We will study this in further depth now.

Exercise 1.3: Coulomb's Law

Let us start out by comparing the two laws by magnitude. The numbers you need are found in the "Useful constants" section 3 on the following page.

a) Estimate the ratio between the gravitational and electrical attraction between a proton and an electron, F_g/F_e.

Answer:

$$\frac{F_g}{F_e} = \frac{Gm_e m_p}{1/\left(4\pi\varepsilon_0\right)e^2} \approx \frac{10^{-11}10^{-27}10^{-31}}{10^{10}(10^{-19})^2} = 10^{-41} \tag{19}$$

[4]A torsion balance is a device that is capable of measuring weak forces. It is based on a spring that works by torsion - twisting it by torque.

[5]This is the same as equation 19.3 in *Lillestøl et. al - Generell Fysikk, bind 2.*

b) Suppose you were able to set up an experiment that measured the acceleration of an electron in the presence of a proton. What would the theoretical difference in acceleration be if you did not include the effect of gravitation?

Answer: Since the gravitational force between them is so small compared to the electric force and since acceleration is proportional to the total force, the difference in acceleration will be negligible.

c) Let's say that the electron is orbiting the proton in a radius of $r = 0.53 \times 10^{-10}$ m. What would the velocity of the electron be if it moves in a perfect circle?

Answer:

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2} \tag{20}$$

$$a = \frac{1}{4\pi\varepsilon_0 m_e}\frac{q_1 q_2}{\mathbf{r}^2} = \frac{v^2}{r} \tag{21}$$

$$v = \frac{1}{4\pi\varepsilon_0 m_e}\frac{e^2}{\mathbf{r}} = 2.19 \times 10^6 \text{ m/s} \tag{22}$$

d) If the electrical force is so much stronger then the gravitational force, why isn't every daily observable phenomenon electrical in nature?

Answer: Since most electrical phenomena occur with charges in pairs, the effect is not as easily visible on a macroscopic scale. Most of the things we observe are neutral in charge.

Useful constants

These constants might be useful in some of this week's exercises.

Electrical permittivity in vacuum:	$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2N^{-1}m^{-2}}$
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{m^2kg^{-1}s^{-2}}$
Proton mass:	$m_p = 1.67 \times 10^{-27}\,\mathrm{kg}$
Proton charge:	$q_p = -1e = -1.602 \times 10^{-19}\,\mathrm{C}$
Electron mass:	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
Electron charge:	$q_e = -1e = -1.602 \times 10^{-19}\,\mathrm{C}$

References