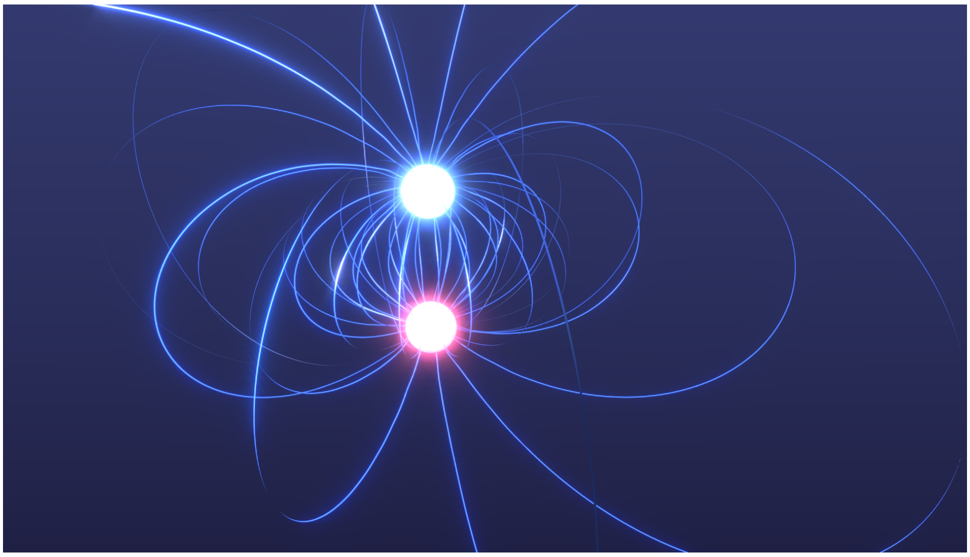
Week 2 – Electric Charges



Nothing is too wonderful to be true if it be consistent with the laws of nature.

Michael Faraday

This week we're diving further into the area of electric charges and fields. We'll for the first time meet what is called a dipole, an important configuration of charges which plays a crucial role in electromagnetic theory. But first we'll explore the concept of the curl of a vector field.

Exercise 2.1: Line integrals and Curl

You've now gotten some practice on calculating the divergence of a vector field, $\nabla \cdot \mathbf{F}$ which returns a *scalar function*. The curl of a vector field, $\nabla \times \mathbf{F}$ is itself a vector field. This is a measure of the *rotation* in the field. Find $\nabla \times \mathbf{F}$ of the following fields.

a) $\mathbf{F} = y\hat{\mathbf{i}}$

Solution:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} && (1)\\
&= \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}0\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}y - \frac{\partial}{\partial x}0\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}y\right)\hat{\mathbf{k}} && (2)\\
&= -\hat{\mathbf{k}} && (3)
\end{aligned}$$

Answer:

$$\nabla \times \mathbf{F} = -\hat{\mathbf{k}}$$

b) $\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$

Solution:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} && (4) \\
&= \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(-x)\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}y - \frac{\partial}{\partial x}0\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}y\right)\hat{\mathbf{k}} && (5) \\
&= -2\hat{\mathbf{k}} && (6)
\end{aligned}$$

Answer:

$$\boldsymbol{\nabla} \times \mathbf{F} = -2\hat{\mathbf{k}}$$

c) $\mathbf{F} = \boldsymbol{\nabla} V$. That is, find the curl of a field which is the gradient of a potential V.

Hint: Start from the definition of the gradient of V.

Solution:

$$\mathbf{F} = \boldsymbol{\nabla} V = \frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

such that

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} && (7) \\
&= \left(\frac{\partial}{\partial y}\frac{\partial V}{\partial z} - \frac{\partial}{\partial z}\frac{\partial V}{\partial y}\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}\frac{\partial V}{\partial x} - \frac{\partial}{\partial x}\frac{\partial V}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial}{\partial y}\frac{\partial V}{\partial x}\right)\hat{\mathbf{k}} && (8) \\
&= 0 && (9)
\end{aligned}$$

Answer:

$$\boldsymbol{\nabla} \times \mathbf{F} = 0$$

Check your results on the fields in (a) and (b) with your intuition[1]. You might want to plot or make a rough sketch of the fields. Again two dimensions will do.

Solution:

[1] Ask yourself whether and which way a tiny paddle wheel centered at the point in question would rotate.

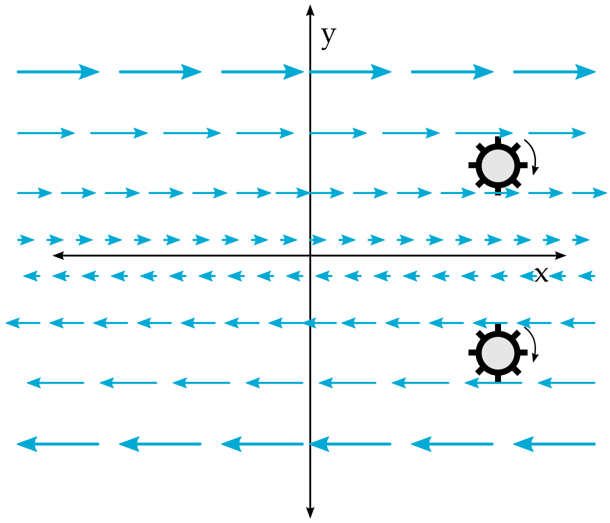


Figure 1: Placing a paddle wheel anywhere in this field (a) would cause it to rotate clockwise because the field gets stronger as we increase y. Thus in the first and second quadrant a paddle wheel would feel a stronger push at higher y-values. As we see from the figure this also holds in the third and fourth quadrant agreeing with the result that the curl points in the negative z-direction (right hand rule).

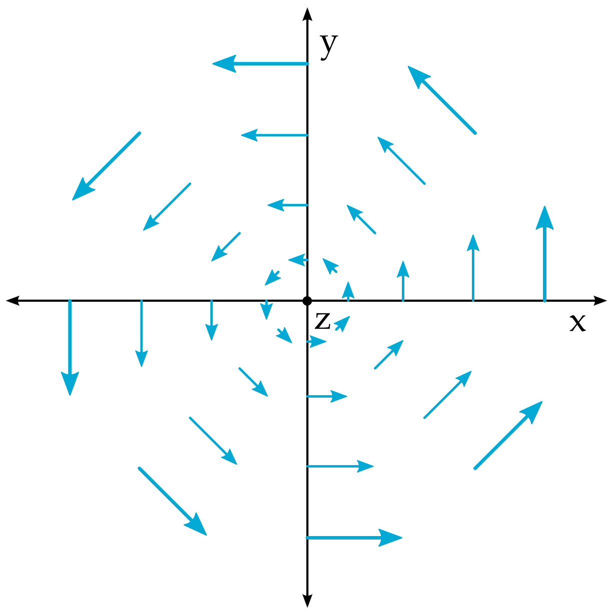


Figure 2: A paddle wheel would also rotate in this field (b), but counter clockwise.

An important field in magnetism has the form $\mathbf{F} = \hat{\boldsymbol{\phi}}/r$ where r is the distance away from the z-axis and $\hat{\boldsymbol{\phi}}$ is a unit vector pointing in the direction of the azimuthal angle. In Cartesian coordinates $r = \sqrt{x^2 + y^2}$ and

$$\hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}},$$

such that

$$\mathbf{F} = \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2 + y^2}.$$

d) Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for a circle of radius r parallel with the xy-plane and centered at the z-axis.

Solution:

A possible parametrization of the circle is

$$\mathbf{r}(t) = r\left(\cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}}\right),$$

for $0 \leq t \leq 2\pi$ such that

$$\mathbf{F}(t) = \frac{r\left(-\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}\right)}{r^2\left(\cos^2 t + \sin^2 t\right)} \tag{10}$$

$$= \frac{-\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}}{r} \tag{11}$$

and

$$\mathbf{r}'(t) = r\left(-\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}\right),$$

giving

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \sin^2 t + \cos^2 t dt \tag{12}$$

$$= 2\pi. \tag{13}$$

Alternatively observe that both $\mathbf{F}$ and $d\mathbf{r}$ is tangent to the circle such that $\mathbf{F} \cdot d\mathbf{r} = |F|dr$ at every point along the circle. Giving

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = |F| \oint_C dr \tag{14}$$

$$= |F|2\pi r \tag{15}$$

$$= 2\pi. \tag{16}$$

Answer:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

e) Show that the curl of **F** is zero everywhere except at the z-axis.

Solution:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} \tag{17}$$

$$= \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}0\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\hat{\mathbf{k}}$$

$$= \left(\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}\right)\hat{\mathbf{k}} \tag{18}$$

$$= \frac{2(x^2+y^2)-2x^2-2y^2}{(x^2+y^2)^2}\hat{\mathbf{k}} \tag{19}$$

$$= 0, \tag{20}$$

as long as $x, y \neq 0$.

f) Use the result from (e) and Stokes' theorem to find $\oint \mathbf{F} \cdot d\mathbf{r}$ for any closed loop which is does not circulate the z-axis.

Solution: Stokes' Theorem states

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{A}. \tag{21}$$

We have that $\nabla \times \mathbf{F} = 0$ everywhere except at the z-axis. This gives us that

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} \mathbf{0} \cdot d\mathbf{A} \tag{22}$$

$$= 0, \tag{23}$$

for any closed curve $\mathcal{C}$ that does not circulate the z-axis.

Exercise 2.2: Electromagnetic Nuts

a) Imagine two spheres of metal hanging side by side. You observe that the spheres attract. From what you know so far, what can you say about the charges the spheres have? Explain what will happen if the spheres touch. Is it possible that they stick together?

Answer: The spheres have opposite charges. Metal is a conductor, so if they have equal charge, they will neutralize each other. If the charge is not equal, the remaining charge will probably be shared equally between the two and they will instead repulse each other. They will not stick together (unless you place something insulating in-between.)

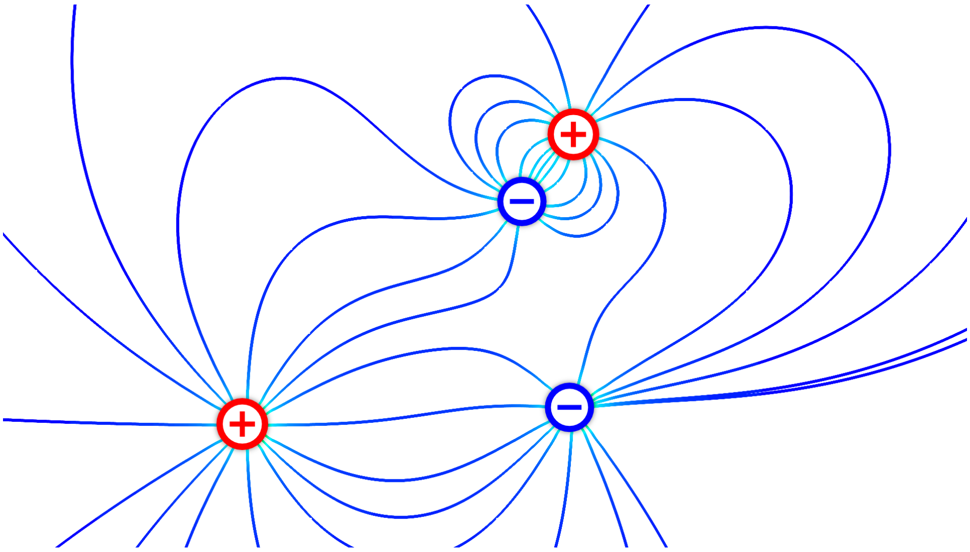


Figure 3: Electric field around four charges in two dimensions.

b) A single electron does not hold much charge alone. But how much charge does a visible object contain? Let's study the electrons contained within a glass marble made singly out of SiO_2. We assume that the marble is 1 cm in radius, weighs 1 g and use that SiO_2 has a molar weight of $60.08\,\mathrm{g}\cdot\mathrm{mol}^{-1}$. Each silicon atom contributes 14 electrons, while each oxygen atom contributes with 8 electrons.[2] How much charge would this add up to in total?

Solution: Divide the marble weight on the molar weight to get the number of mole molecules.

$$n_m = \frac{1\,\mathrm{g}}{60.08\,\mathrm{g}\cdot\mathrm{mol}^{-1}} = 0.017\,\mathrm{mol}$$

Multiply this with $14 + 8 + 8 = 30$ electrons to get the number of mole electrons.

$$n_e = 30 \cdot 0.017\,\mathrm{mol} = 0.5\,\mathrm{mol}$$

Multiply this with Avogadro's number,

$$n_e = 0.5\,\mathrm{mol} \cdot 6.022 \cdot 10^{23}\,\mathrm{mol}^{-1} = 3.011 \cdot 10^{23}$$

Multiply this with the electron charge,

$$Q = 1.602 \cdot 10^{-19}\,\mathrm{C} \cdot 3.011 \cdot 10^{23}$$

to get

Answer: $Q \approx 48\,180\,\mathrm{C}$.

Exercise 2.3: Field from a dipole

Field lines are a great way to visualize the electric field around charges. They convey a lot of information and gives an intuitive way to picture the field. In figure 3 you can see a fairly complicated field from two positive and two negative charges.

[2]The electrons per shell are 2, 8, 4 for Si and 2, 6 for O.

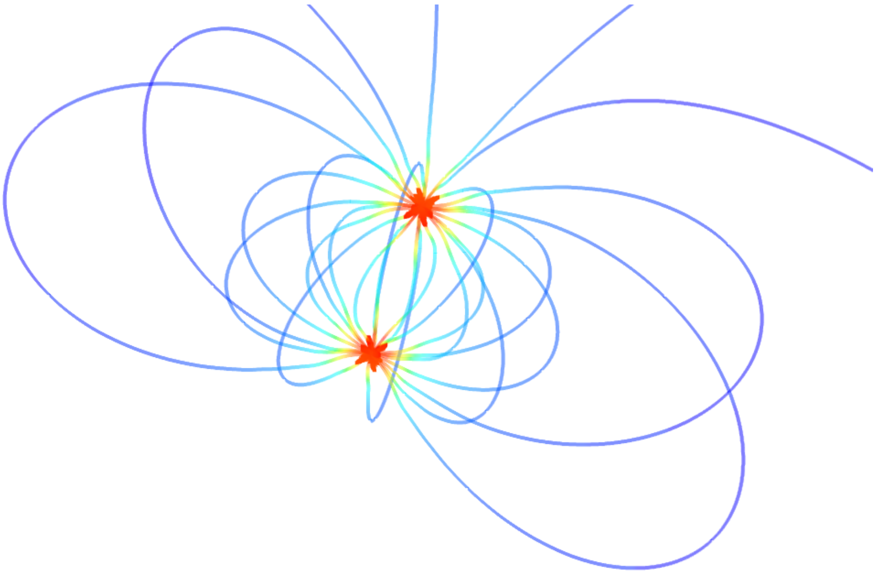


Figure 4: Electric field around a dipole in 3D.

In this exercise we will study the field of a dipole. Consider a charge $q_1 = -q$ located at $(-d, 0, 0)$ and a positive charge $q_2 = q$ located at $(d, 0, 0)$. Both charges are held fixed (they are not moving).

Some of these exercises require you to run applications in Python and Mayavi. If you have not done so already, you should have a look at our Getting Started With Python *guide that you can find on our webpages.*[3]

a) Make a sketch of the electric field from the dipole in two dimensions.

b) In this week's code folder, you'll find a program named *dipole.py*. Run this in IPython by typing the following commands in a terminal:[4]

```
ipython --pylab  ## use ipython -wthread on Ubuntu
%run dipole.py
```

Compare the visualization with your sketch of the dipole.

Note that the visualization will contain some field lines that end up in nowhere. This has no physical meaning (electric field lines don't stop in empty space) and is only because the field we calculate has a limited size.

c) Even though a two-dimensional representation of an electric field often is good enough, these fields really do live in three dimensions. Run the program *dipole3d.py* to visualize a three dimensional version of the dipole field. Use the left mouse button to rotate the view. You can also have a look at figure 4 if you are unable to use the program.

Let's now dive into some calculations on the dipole. The situation is shown in figure 5. We still have $q_1 = -q$ and $q_2 = q$.

d) Show that the electric field at points on the x-axis where $|x| > d$ can be written as

$$E_x = \frac{1}{2\pi\varepsilon_0} \frac{p|x|}{(x^2 - d^2)^2}$$
$$E_y = 0$$

[3] The webpages are located at http://mindseye.no/fys1120/notes/

[4] Change your working directory to the folder where you saved the files using the *cd* command in a terminal. For instance, *cd /home/lasse/Downloads/week2/*

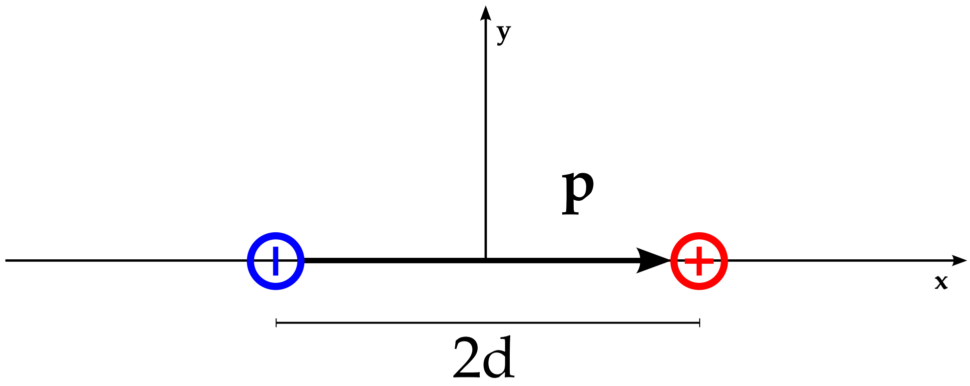


Figure 5: A dipole centered at the origin with $\mathbf{p}$ along the x-axis.

$$E_z = 0,$$

where $\mathbf{p} = 2dq\hat{\mathbf{i}}$ is the dipole moment. Show that when $x \gg d$ E_x will be inversely proportional to x^3.

Solution: We first solve the problem for $x > d$. The field from the negative charge is denoted $\mathbf{E}_-$ and the field from the positive $\mathbf{E}_+$. For simplicity we define $k = 1/(4\pi\varepsilon_0)$.

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_- + \mathbf{E}_+ && (24)\\
&= -k\frac{q}{(x+d)^2}\hat{\mathbf{i}} + k\frac{q}{(x-d)^2}\hat{\mathbf{i}} && (25)\\
&= kq\frac{(x+d)^2-(x-d)^2}{(x-d)^2(x+d)^2}\hat{\mathbf{i}} && (26)\\
&= kq\frac{4xd}{(x^2-d^2)^2}\hat{\mathbf{i}} && (27)\\
&= \frac{1}{4\pi\varepsilon_0}\frac{4xqd}{(x^2-d^2)^2}\hat{\mathbf{i}} && (28)\\
&= \frac{1}{2\pi\varepsilon_0}\frac{x(2qd\hat{\mathbf{i}})}{(x^2-d^2)^2} && (29)\\
&= \frac{1}{2\pi\varepsilon_0}\frac{x\mathbf{p}}{(x^2-d^2)^2} && (30)\\
& && (31)
\end{aligned}$$

Which means that the y and z components of the field must be zero. By looking at the symmetry, we see that the solution for $x < -d$ must be of equal magnitude (but opposite in direction).

When $x \gg d$, meaning that we are far away from the dipole along the x-axis, d becomes negligible compared to x. We then have $(x^2 - d^2)^2 \approx x^4$ such that

$$\frac{1}{2\pi\varepsilon_0}\frac{xp}{(x^2-d^2)^2} \approx \frac{1}{2\pi\varepsilon_0}\frac{p}{x^3} \qquad (32)$$

Which means that the field strength goes down more rapidly for a dipole than for a point charge.

Answer: When $x \gg d$

$$E_x \approx \frac{1}{2\pi\varepsilon_0} \frac{p}{x^3}$$

e) Can you find any points along the x-axis where the field is exactly zero?

Answer: Nowhere, but the strength of the field goes towards zero as we move far away from the dipole.

f) Now suppose the charges weren't exactly equal in magnitude. That is $|q_1| \neq |q_2|$. Will there now be any points on the x-axis where the field is zero?

Hint: Analyze the field in three regions, $x > d$, $x < -d$ and $-d < x < d$. You don't need to do any calculations.

Answer: There is one point along the x-axis where the field from the bigger charge will exactly cancel the field of the smaller one. This will happen in the region right next to the smaller charge.

g) Let the charges have equal magnitude such that our configuration again is identical to the dipole configuration of figure 5. Find the field along the y-axis and show that for $y \gg d$ it will proportional to the inverse of y^3.

Hint: How will the vectors add along the y-axis? Make a sketch.

Answer:

The field will only point in the negative x-direction and

$$E_x = -\frac{1}{4\pi\varepsilon_0} \frac{p}{(y^2 + d^2)^{3/2}} \approx -\frac{1}{4\pi\varepsilon_0} \frac{p}{y^3}$$

for $y \gg d$.

Solution:

Referring to figure 6 we see that the y-components of the field will cancel for points along the y-axis. We also see that it will point in the negative x-direction. This gives

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_- + \mathbf{E}_+ &(33)\\
&= -\left(E_- \cos\theta + E_+ \cos\theta\right) \hat{\mathbf{i}} &(34)\\
&= -\left(E_- + E_+\right) \cos\theta \hat{\mathbf{i}} &(35)\\
&= -\left(\frac{kq}{(y^2 + d^2)} + \frac{kq}{(y^2 + d^2)} \frac{d}{\sqrt{y^2 + d^2}}\right) \hat{\mathbf{i}} &(36)\\
&= -\frac{k(2qd\hat{\mathbf{i}})}{(y^2 + d^2)^{3/2}} &(37)\\
&= -\frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p}}{(y^2 + d^2)^{3/2}}, &(38)
\end{aligned}$$

and for $y \gg d$

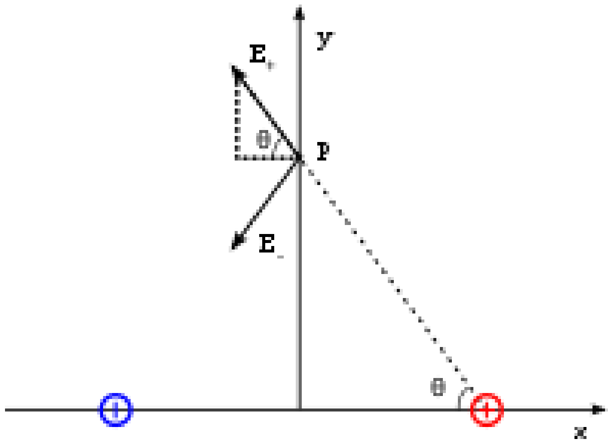


Figure 6: Because of the symmetry across the y-axis the y-components of hte fields will exactly cancel for points on the y-axis.

$$-\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{(y^2+d^2)^{3/2}} \approx -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{y^3}.$$

Useful constants

These constants might be useful in some of this week's exercises.

Electrical permittivity in vacuum:	$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2N^{-1}m^{-2}}$
Gravitational constant:	$G = 6.67 \times 10^{-11}\,\mathrm{m^2kg^{-1}s^{-2}}$
Proton mass:	$m_p = 1.67 \times 10^{-27}\,\mathrm{kg}$
Proton charge:	$q_p = 1e = 1.602 \times 10^{-19}\,\mathrm{C}$
Electron mass:	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
Electron charge:	$q_e = -1e = -1.602 \times 10^{-19}\,\mathrm{C}$