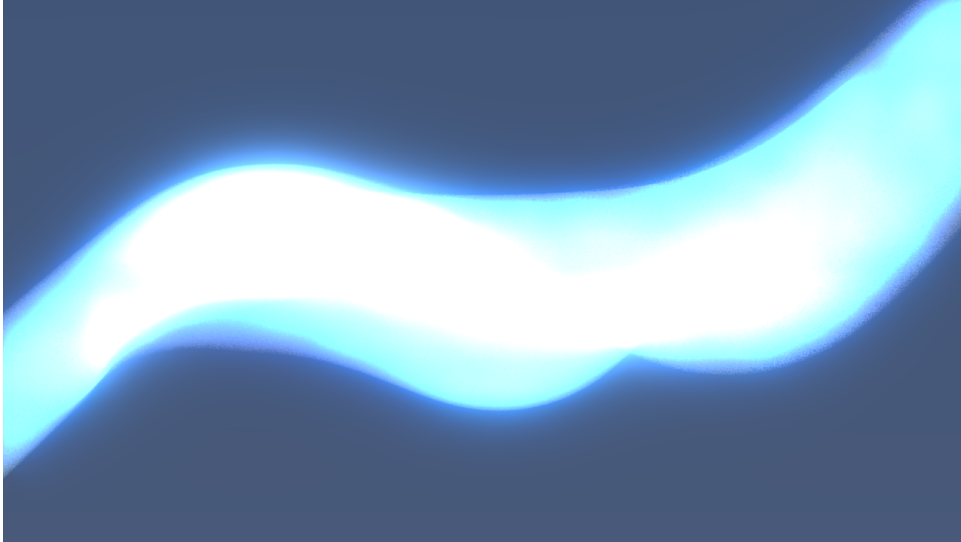


## Week 3 – Continuous charge have a lot of potential



The world is continuous, but the mind is discrete.

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### Exercise 3.1: Field from continuous charge distributions

When we have continuous charge distributions the usual sum over point charges must be replaced by an integral

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dq,$$

where  $|\mathbf{r} - \mathbf{r}'|$  is the distance from the infinitesimal charge source to the point  $\mathbf{r}$  with charge  $dq$ . Furthermore  $dq = \lambda dx$  for a line,  $dq = \sigma da$  for a surface and  $dq = \rho dv$  for a volume<sup>1</sup>.

Here we will study the electric field from a rod placed along the x-axis of length  $2L$  and with charge density  $\lambda$ . The situation is shown in figure 1.

- a) Determine the direction of the field in the point  $P$ , a distance  $z$  normal to the middle of the rod (on the  $z$ -axis). Show that the strength of the electric field can be written as

$$E_z = \frac{2\lambda z}{4\pi\epsilon_0} \int_0^L \frac{1}{(x^2 + z^2)^{3/2}} dx.$$

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<sup>1</sup>You might be more used to the notation  $dV$  for a volume element, but we use a little  $v$  in  $dv$  to distinguish from the electrostatic potential  $V$

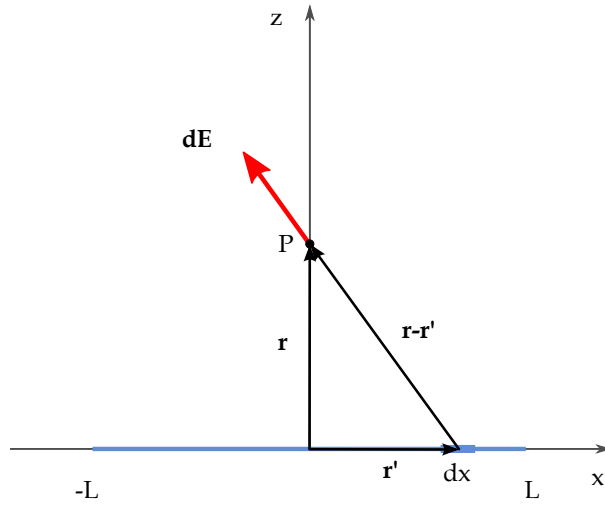


Figure 1: A finite line segment. Notice the symmetry across the z-axis.

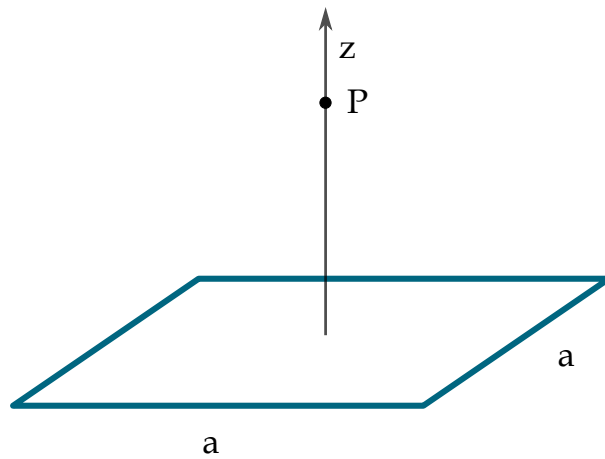


Figure 2: We can use the field from a finite rod to find the field from a square loop.

where  $E_x = E_y = 0$ . *Hint: Pair up one contribution at  $-x$  with another at  $x$ . Is there some cancellation going on here?*

- b) Evaluate the integral. You might want to use the substitution  $x = z \tan \theta$ .

*Answer:*

$$E_z = \frac{2\lambda z}{4\pi\epsilon_0} \frac{1}{z^2} \frac{L}{\sqrt{L^2 + z^2}}$$

- c) What do you expect the field to look like when you move very far away from the rod, i.e. for  $z \gg L$ ? Does the expression you have for the field make sense in this limit?

*Answer:* Looks like the field from a point charge.

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{2L\lambda}{z^2}$$

- d) What does the field look like for an infinite rod,  $L \rightarrow \infty$ ? The expression you found in (b) was only valid along the  $z$ -axis. Where is this expression valid?
- e) Use your result from (b) to find the field at the point  $P$  a distance  $z$  from the center of a square loop with sides of length  $a$  (figure 2).

*Hint: Note that the  $z$ -axis in (b) does not correspond to the  $z$ -axis in figure 2. What is  $z$  and  $L$  in this case?*

### Exercise 3.2: Potentials, voltmeters and static electricity

- a) If the electric field is zero in a region of space, what value is the electric potential? Is it also zero?

*Answer:* No. The potential is always relative to another point, and so it can inherit any value without regard to the absolute value of the electric field.

- b) In circuits we often use a voltmeter (voltage being just another word for the electric potential) to measure potential. How can the voltmeter know what to read when the potential is a value relative to any reference point  $\mathcal{O}$ ?

*Answer:* It reads out the potential difference between the two terminals on the voltmeter. Thus, it does not read out the potential, but the potential *difference*.

- c) If you shuffle your shoes across a nylon carpet you can obtain a potential difference between yourself and the floor of several thousand volts. If you then touch a metal sink you might feel a mild shock, but touching a power line of the same voltage could be lethal. Why doesn't touching the sink kill you?

*Answer:* Even if there is a high potential difference, it might not carry enough energy to kill you. The power line is lethal because it does not only have a high potential difference, but also a huge amount of current ready to push through your body.

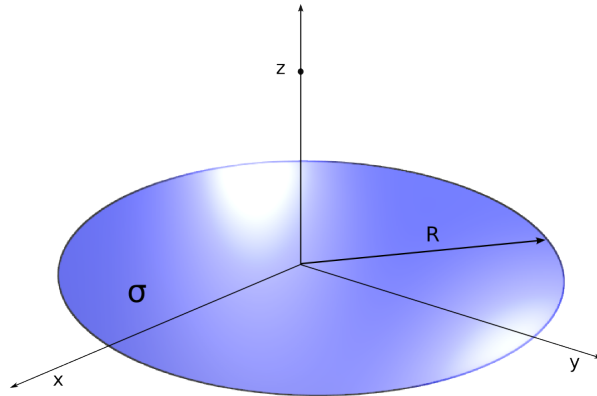


Figure 3: Disk of radius  $R$ .

### Exercise 3.3: Electrostatic potential

One fundamental property of electric fields is that  $\nabla \times \mathbf{E} = 0$  which allows us to define a potential

$$V(\mathbf{r}) \equiv - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r},$$

where  $\mathcal{O}$  is some reference point which are often chosen to be at infinity. When  $\mathcal{O}$  is at infinity one finds the potential for a discrete set of charges by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}$$

and for the continuous case we have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq'}{|\mathbf{r} - \mathbf{r}'|},$$

where  $dq' = \rho(r')dv'$ .

The electric field can then be found by the relation

$$\mathbf{E} = -\nabla V.$$

Because the potential is a scalar function it is usually much simpler to find by integration than the electric field itself. Therefore the potential, along with numerous other applications, can serve as a simplifying intermediate step in finding the electric field. We'll study this property in the next exercises.

- a) Find the potential from a disk of radius  $R$  with uniform surface charge  $\sigma$  a distance  $z$  above the center of the disc and show that it can be written as

$$V(z) = \frac{\sigma}{2\epsilon_0} \left( \sqrt{R^2 + z^2} - z \right)$$

The situation is shown in figure 3.

- b) Compute the electric field.  
 c) Check the limit when  $z \gg R$  and find the field for an infinite disc by letting  $R \rightarrow \infty$ .

In the case of the infinite disc, how does the field vary with distance? How can you justify this result?

## Exercise 3.4: Plotting the field from an uniformly charged sphere

In the next exercise you will be asked to find the field from an uniformly charged solid sphere, but first; let's have a look at the field and play around with its properties.

To find the field, it is often easier to find the potential first. As you will see in the next exercise, the field inside an uniformly charged sphere is

$$V(\mathbf{r}) = \frac{Q}{8\pi\epsilon_0 R} \left( 3 - \frac{r^2}{R^2} \right)$$

while outside the sphere, it is

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r}.$$

- a) Use the `contour3d()` function in Mayavi to visualize the field in 3D. Set the number of contours to 20 and the opacity to 0.5 to be able to see through the contours (see the code in the hint below on how to do this).

*Hint: Use the following code to find the distance to each point in space. You need to use the distance to calculate the potential.*

```
from numpy import *
x,y,z = mgrid[-100.:101.:5., -100.:101.:5., -100.:101.:5.]
r = sqrt(x**2 + y**2 + z**2)
V = 0*x
for i in range(len(r)):
    for j in range(len(r)):
        for k in range(len(r)):
            # This prints out the distance to the point in question.
            # Replace it with your calculation of the potential.
            print r[i][j][k]
            V[i][j][k] = ...
...
contour3d(x, y, z, V, contours=20, opacity=0.5)
```

- b) Use the Numpy function `gradient()` to find the electric field as  $\mathbf{E} = -\nabla V$ . Plot the electric field using the `quiver3d()` function.

See the course pages for information on how to set up and use Python and Mayavi.

## Exercise 3.5: Finding the field from an uniformly charged solid sphere

As you already know Newton's universal law of gravitation is very similar to Coulomb's law and the electric field. The force laws give us the force of attraction between two point masses or point charges separated by a distance  $r$  in space. However there is really no such thing as a *point* mass/charge, especially not when you're formulating laws for the attraction between the earth and the sun. So Newton had to prove that the force between planets were as if all their mass was located at the center.

In other words he had to prove the equivalence between spherical mass distributions and that of a point mass<sup>2</sup>. Here we will do the same thing for charge distributions and we will do it by using the potential<sup>3</sup>.

<sup>2</sup>Newton claimed he discovered the law of gravitation in his early years, but held back the publication for a long time because he could not prove the equivalence between spherical and point mass distributions. He first had to invent the calculus.

<sup>3</sup>Proving it by direct integration of the field is challenging, but absolutely doable.

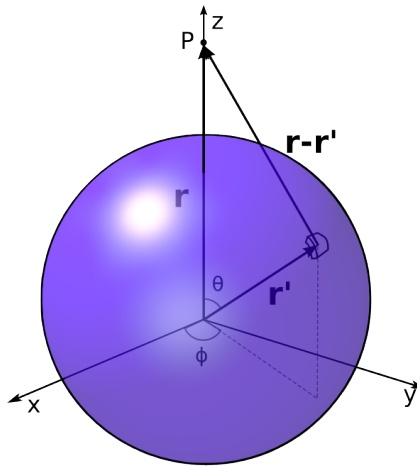


Figure 4: A sphere of uniform charge density.

- a) The situation is shown in figure 4. The sphere has a radius  $R$  and is *uniformly* charged with a charge density  $\rho$ . The distance from a contributing volume element  $dv$  is now  $|\mathbf{r} - \mathbf{r}'|$ , but this distance varies with  $\theta$  so we need to relate the two. Use the law of cosines to show that  $|\mathbf{r} - \mathbf{r}'|^2 = r^2 + r'^2 - 2rr' \cos \theta$  and use this to show that the potential can be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho r'^2 \sin \theta d\phi d\theta dr'}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}},$$

where this is evaluated as a triple integral. What are the limits of the integral?

- b) Evaluate the integral and find the potential *outside* the sphere. Show that it can be written as

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r}.$$

*Hint: A simple substitution might be good here. Hint:  $\sqrt{r^2 + r'^2 - 2rr'} = \sqrt{(r - r')^2} = |r - r'|$*

- c) Find the electric field from the sphere and compare it with the electric field from a point charge.

*Answer:*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$$

- d) Go a few steps back in your derivation above and find out what's different when we want to find the potential *inside* the sphere. Show that the potential here can be written as

$$V(\mathbf{r}) = \frac{Q}{8\pi\epsilon_0 R} \left( 3 - \frac{r^2}{R^2} \right)$$

and find the electric field here. Is the field continuous in  $r = R$ ?

*Answer:*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}}$$