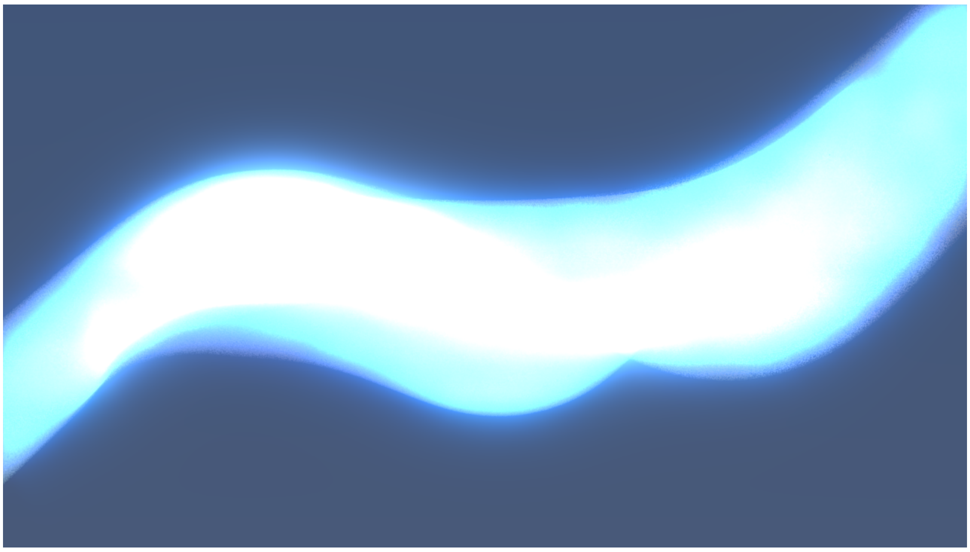
Week 3 – Continuous charge have a lot of potential



The world is continuous, but the mind is discrete.

David Mumford

Exercise 3.1: Field from continuous charge distributions

When we have continuous charge distributions the usual sum over point charges must be replaced by an integral

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}dq,$$

where $|\mathbf{r}-\mathbf{r}'|$ is the distance from the infinitesimal charge source to the point $\mathbf{r}$ with charge dq. Furthermore $dq = \lambda\ dx$ for a line, $dq = \sigma\ da$ for a surface and $dq = \rho\ dv$ for a volume[1].

Here we will study the electric field from a rod placed along the x-axis of length $2L$ and with charge density λ. The situation is shown in figure 1.

a) Determine the direction of the field in the point P, a distance z normal to the middle of the rod (on the z-axis). Show that the strength of the electric field can be written as

$$E_z = \frac{2\lambda z}{4\pi\varepsilon_0}\int_0^L \frac{1}{(x^2+z^2)^{3/2}}dx.$$

[1] You might be more used to the notation dV for a volume element, but we use a little v in dv to distinguish from the electrostatic potential V

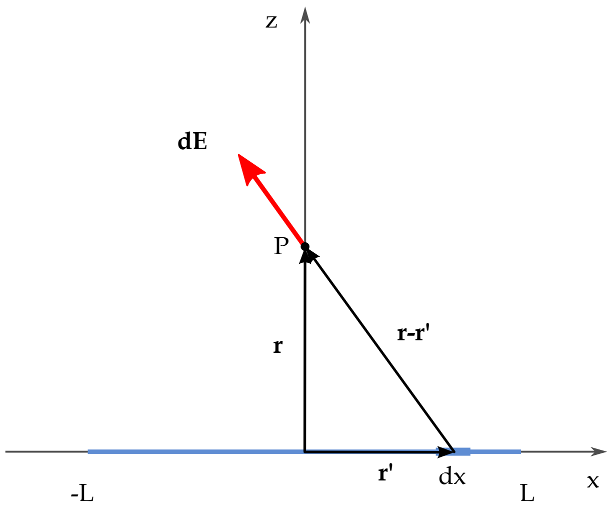


Figure 1: A finite line segment. Notice the symmetry across the z-axis.

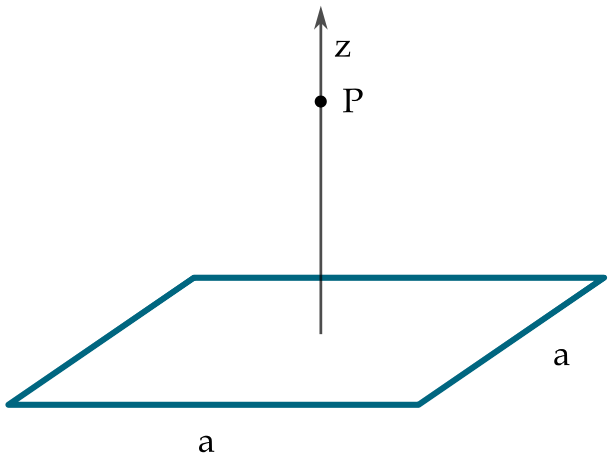


Figure 2: We can use the field from a finite rod to find the field from a square loop.

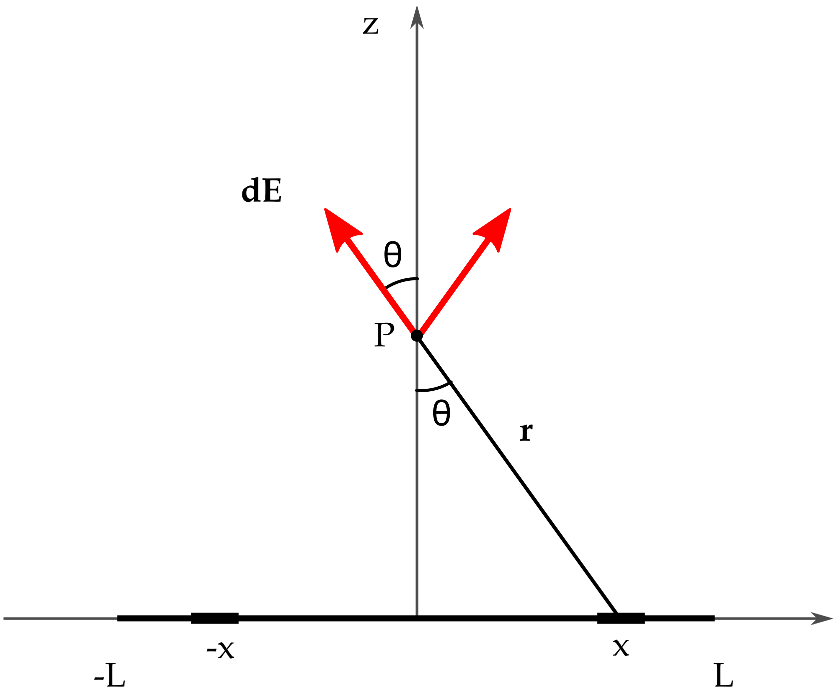


Figure 3: Pairing up contributions at x and at $-x$ it becomes clear that only the z-component of the field survives.

where $E_x = E_y = 0$. *Hint: Pair up one contribution at $-x$ with another at x. Is there some cancellation going on here?*

Solution:

Pairing up contributions from a little charge dq at x and another one at -x we see that the all the components of the field except for the z-component will cancel. The z-component from a little charge element at x is given by

$$dE_z = |d\mathbf{E}| \cos\theta = \frac{1}{4\pi\varepsilon_0} \frac{dq}{x^2 + z^2} \frac{z}{\sqrt{x^2 + z^2}},$$

where θ is the angle between the vector $d\mathbf{E}$ and the z-axis and $dq = \lambda dx$. The total field at P is therefore

$$E_z = \frac{1}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{dq}{(x^2 + z^2)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{dx}{(x^2 + z^2)^{3/2}}$$

and because the contributions at the negative part of the x-axis is equal to those at the positive part (mathematically because $1/(x^2 + z^2)^{3/2}$ is an even function) this can be written as

$$E_z = \frac{2\lambda}{4\pi\varepsilon_0} \int_{0}^{L} \frac{dx}{(x^2 + z^2)^{3/2}}.$$

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b) Evaluate the integral. You might want to use the substitution $x = z\tan\theta$.

Solution:

$$x = z\tan\theta \Rightarrow dx = z\frac{1}{\cos^2\theta}d\theta$$

$$(x^2+z^2)^{3/2} = z^3(\tan\theta+1)^{3/2} = z^3\frac{1}{\cos^3\theta}$$

such that

$$\begin{aligned}
\int \frac{dx}{(x^2+z^2)^{3/2}} &= \frac{1}{z^2}\int \frac{1}{\cos^2\theta}\cos^3\theta d\theta && (1)\\
&= \frac{1}{z^2}\int \cos\theta d\theta && (2)\\
&= \frac{1}{z^2}\sin\theta && (3)\\
&= \frac{1}{z^2}\frac{x}{\sqrt{x^2+z^2}}. && (4)
\end{aligned}$$

Therefore

$$E_z = \frac{2\lambda z}{4\pi\varepsilon_0}\int_0^L \frac{dx}{(x^2+z^2)^{3/2}} = \frac{2\lambda z}{4\pi\varepsilon_0}\frac{1}{z^2}\left[\frac{x}{(x^2+z^2)^{3/2}}\right]_0^L = \frac{2\lambda z}{4\pi\varepsilon_0}\frac{1}{z^2}\frac{L}{\sqrt{L^2+z^2}}$$

where $\sin\theta$ can be expressed in terms of x by looking at the triangle in figure 3

Answer:

$$E_z = \frac{2\lambda z}{4\pi\varepsilon_0}\frac{1}{z^2}\frac{L}{\sqrt{L^2+z^2}}$$

c) What do you expect the field to look like when you move very far away from the rod, i.e. for $z \gg L$? Does the expression you have for the field make sense in this limit?

Solution:

Everything looks like a point when we're far enough away, so the field from the rod should look like the field from a point charge. When $z \gg L$, L is negligible compared to z such that $\sqrt{L^2+z^2} \approx L$ and

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{L^2+z^2}} \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}$$

where $2L\lambda = Q$ is the total charge on the rod. Indeed, this looks like the field from a point charge.

Answer: Looks like the field from a point charge.

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{2L\lambda}{z^2}$$

d) What does the field look like for an infinite rod, $L \to \infty$? The expression you found in (b) was only valid along the z-axis. Where is this expression valid?

Solution: For an infinite rod we take the limit $L \to \infty$ of E_z.

$$\lim_{L\to\infty} E_z = \lim_{L\to\infty} \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z\sqrt{1+\left(\frac{z}{L}\right)^2}} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}.$$

For a finite rod we could exploit the fact that on the axes normal to the *middle* of the rod the contributions from equidistant charges dq would cancel the x-component of the total field. Now we have an infinite rod which means that we'll always be able to pair up equidistant contributions having the same cancelling effect. So in comparison with the finite rod where we could only use this argument along the axis normal to the middle of the rod, this is now true everywhere. This means that for an infinite rod the field

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}\hat{\mathbf{r}},$$

where r is the distance from the rod in cylindrical coordinates, is valid everywhere.

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e) Use your result from (b) to find the field at the point P a distance z from the center of a square loop with sides of length a (figure 2).

Hint: Note that the z-axis in (b) does not correspond to the z-axis in figure 2. What is z and L in this case?

Solution:

From (b) we had that

$$E_z = \frac{2\lambda z}{4\pi\varepsilon_0}\frac{1}{z^2}\frac{L}{\sqrt{L^2+z^2}}.$$

Here $L \to a/2$ and $z \to \sqrt{z^2+(\frac{a}{2})^2}$ so that the field from one side from the square loop in point P is

$$\mathbf{E}_1 = \frac{2\lambda\sqrt{z^2+(\frac{a}{2})^2}}{4\pi\varepsilon_0}\frac{\frac{a}{2}}{\left(\left(\frac{a}{2}\right)^2+z^2\right)\sqrt{\left(\frac{a}{2}\right)^2+z^2+\left(\frac{a}{2}\right)^2}}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda a}{\sqrt{z^2+\frac{a^2}{2}}\sqrt{\left(\frac{a}{2}\right)^2+z^2}}\hat{\mathbf{r}},$$

where the field points radially away as show in figure 4. Now there are 4 sides in total and the field from the other 3 sides will cancel the component parallel to the plane of the square, so we get the total field by multiplying by $4\cos\theta$. Expressed in terms of the lengths in the problem this becomes

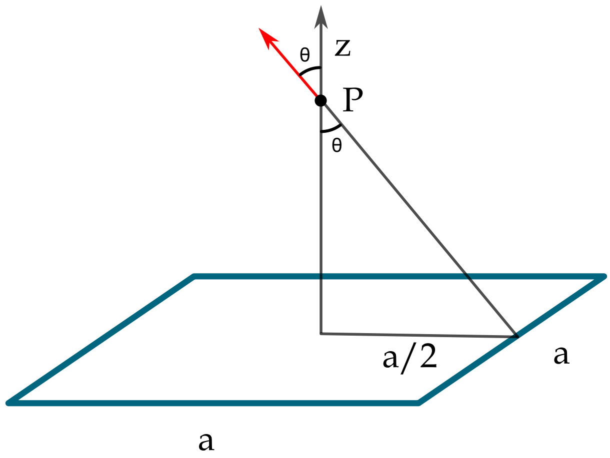

Figure 4

$$\mathbf{E}_{tot} = 4\frac{1}{4\pi\varepsilon_0}\frac{\lambda a}{\sqrt{z^2+\frac{a^2}{2}}\sqrt{\left(\frac{a}{2}\right)^2+z^2}}\frac{z}{\sqrt{\left(\frac{a}{2}\right)^2+z^2}}\hat{\mathbf{k}} = \frac{1}{4\pi\varepsilon_0}\frac{4\lambda a z}{\left(\left(\frac{a}{2}\right)^2+z^2\right)\sqrt{z^2+\frac{a^2}{2}}}\hat{\mathbf{k}}.$$

Exercise 3.2: Potentials, voltmeters and static electricity

a) If the electric field is zero in a region of space, what value is the electric potential? Is it also zero?

Answer: No. The potential is always relative to another point, and so it can inherit any value without regard to the absolute value of the electric field.

b) In circuits we often use a voltmeter (voltage being just another word for the electric potential) to measure potential. How can the voltmeter know what to read when the potential is a value relative to any reference point $\mathcal{O}$?

Answer: It reads out the potential difference between the two terminals on the voltmeter. Thus, it does not read out the potential, but the potential *difference*.

c) If you shuffle your shoes across a nylon carpet you can obtain a potential difference between yourself and the floor of several thousand volts. If you then touch a metal sink you might feel a mild shock, but touching a power line of the same voltage could be lethal. Why doesn't touching the sink kill you?

Answer: Even if there is a high potential difference, it might not carry enough energy to kill you. The power line is lethal because it does not only have a high potential difference, but also a huge amount of current ready to push through your body.

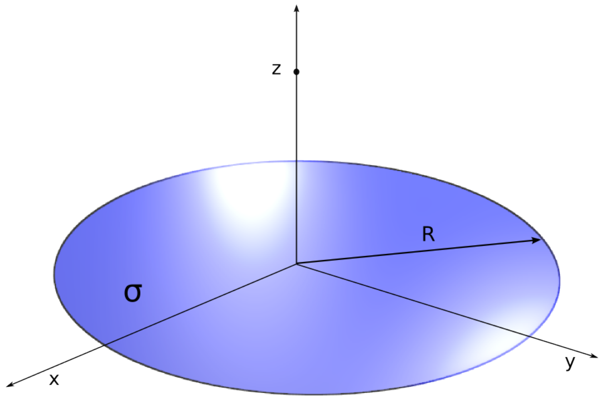

Figure 5: Disk of radius R.

Exercise 3.3: Electrostatic potential

One fundamental property of electric fields is that $\nabla \times \mathbf{E} = 0$ which allows us to define a potential

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{r} \mathbf{E} \cdot d\mathbf{r},$$

where $\mathcal{O}$ is some reference point which are often chosen to be at infinity. When $\mathcal{O}$ is at infinity one finds the potential for a discrete set of charges by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}$$

and for the continuous case we have

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq'}{|\mathbf{r} - \mathbf{r}'|},$$

where $dq' = \rho(r')dv'$.

The electric field can then be found by the relation

$$\mathbf{E} = -\nabla V.$$

Because the potential is a scalar function is it usually much simpler to find by integration than the electric field itself. Therefore the potential, along with numerous other applications, can serve as a simplifying intermediate step in finding the electric field. We'll study this property in the next exercises.

a) Find the potential from a disk of radius R with uniform surface charge σ a distance z above the center of the disc and show that it can be written as

$$V(z) = \frac{\sigma}{2\varepsilon_0} \left(\sqrt{R^2 + z^2} - z \right)$$

The situation is shown in figure 5.

Solution:

Here the distance r to the point where we are trying to find the potential is z, i.e $r = z$.

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq'}{|\mathbf{r}-\mathbf{r}'|}$$

and from the figure 5 we see that the distance from the source charge dq' and the point P is $|\mathbf{r}-\mathbf{r}'| = \sqrt{z^2+r'^2}$. Since we are considering a *surface* $dq' = \sigma da = \sigma r' d\theta dr'$ such that

$$\begin{aligned} V(z) &= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{\sigma r' dr' d\theta}{\sqrt{z^2+r'^2}} && (5)\\ &= \frac{2\pi\sigma}{4\pi\varepsilon_0}\frac{1}{2}\int_{z^2}^{z^2+R^2}\frac{du}{u^{1/2}} && (6)\\ &= \frac{1}{2\sigma}\left(\sqrt{z^2+R^2}-z\right) && (7)\end{aligned}$$

where we used the substitution $u = z^2 + r'^2 \Rightarrow du = 2r'dr'$.

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b) Compute the electric field.

Solution: The electric field is found from the relation $\nabla V = -\mathbf{E}$.

$$\begin{aligned} \nabla V &= \frac{\partial}{\partial x}V\hat{\mathbf{i}} + \frac{\partial}{\partial y}V\hat{\mathbf{j}} + \frac{\partial}{\partial z}V\hat{\mathbf{i}} && (8)\\ &= \frac{\partial}{\partial z}\frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-z\right)\hat{\mathbf{k}} && (9)\\ &= \frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+R^2}}-1\right)\hat{\mathbf{k}} && (10)\\ &= -\mathbf{E} && (11)\end{aligned}$$

so that

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)\hat{\mathbf{k}}$$

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c) Check the limit when $z \gg R$ and find the field for an infinite disc by letting $R \to \infty$.

In the case of the infinite disk, how does the field vary with distance? How can you justify this result?

Solution: One could first try to argue that when $z \gg R$ $\sqrt{z^2+R^2} \approx z$, but then

$$\mathbf{E} \approx \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{z}\right)\hat{\mathbf{k}} = 0\hat{\mathbf{k}}$$

which is certainly true when we get too far away, but before the field vanishes it should look like the field from a point charge. Setting $z/\sqrt{z^2+R^2} \approx 1$ is in fact only keeping the first term in the taylor expansion. Expanding we see that

$$\frac{z}{z\sqrt{1+\left(\frac{R}{z}\right)^2}} = \left(1+\left(\frac{R}{z}\right)^2\right)^{-1/2} = 1-\frac{1}{2}\left(\frac{R}{z}\right)^2 + \dots$$

so by keeping the first two terms we get

$$\mathbf{E} \approx \frac{\sigma}{2\varepsilon_0}\left(1-\left(1-\frac{1}{2}\left(\frac{R}{z}\right)^2\right)\right)\hat{\mathbf{k}} = \frac{\pi R^2\sigma}{4\pi\varepsilon_0 z^2}\hat{\mathbf{k}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}\hat{\mathbf{k}}$$

which is indeed the field from a point charge.

Exercise 3.4: Plotting the field from an uniformly charged sphere

In the next exercise you will be asked to find the field from an uniformly charged solid sphere, but first; let's have a look at the field and play around with its properties.

To find the field, it is often easier to find the potential first. As you will see in the next exercise, the field inside an uniformly charged sphere is

$$V(\mathbf{r}) = \frac{Q}{8\pi\varepsilon_0 R}\left(3-\frac{r^2}{R^2}\right)$$

while outside the sphere, it is

$$V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}.$$

a) Use the **contour3d()** function in Mayavi to visualize the field in 3D. Set the number of contours to 20 and the opacity to 0.5 to be able to see through the contours (see the code in the hint below on how to do this).

   Hint: Use the following code to find the distance to each point in space. You need to use the distance to calculate the potential.

```
from numpy import *
x,y,z = mgrid[-100.:101.:5., -100.:101.:5., -100.:101.:5.]
r = sqrt(x**2 + y**2 + z**2)
V = 0*x
for i in range(len(r)):
    for j in range(len(r)):
        for k in range(len(r)):
            # This prints out the distance to the point in question.
            # Replace it with your calculation of the potential.
            print r[i][j][k]
            V[i][j][k] = ...
...
contour3d(x, y, z, V, contours=20, opacity=0.5)
```

b) Use the Numpy function **gradient()** to find the electric field as $\mathbf{E} = -\nabla V$. Plot the electric field using the **quiver3d()** function.

Solution: An example script:

```
from numpy import *
from enthought.mayavi.mlab import *

x,y,z = mgrid[-100:101:5., -100:101:5., -100:101:5.]
R = 40
Q = 1.0

r = sqrt(x**2 + y**2 + z**2)
V = 0*x
for i in range(len(r)):
    for j in range(len(r)):
        for k in range(len(r)):
            if r[i][j][k] < R:
                V[i][j][k] = Q / (8 * pi * R) * (3 - r[i][j][k]**2/R**2)
            else:
                V[i][j][k] = Q / (4 * pi * r[i][j][k])

contour3d(x, y, z, V, contours=20, opacity=0.5)

Ex,Ey,Ez = gradient(V)
Ex = - Ex
Ey = - Ey
Ez = - Ez

#quiver3d(x, y, z, Ex, Ey, Ez)
```

See the course pages for information on how to set up and use Python and Mayavi.

Exercise 3.5: Finding the field from an uniformly charged solid sphere

As you already know Newton's universal law of gravitation is very similar to Coulomb's law and the electric field. The force laws give us the force of attraction between two point masses or point charges separated by a distance r in space. However there is really no such thing as a *point* mass/charge, especially not when you're formulating laws for the attraction between the earth and the sun. So Newton had to prove that the force between planets were as if all their mass was located at the center.

In other words he had to prove the equivalence between spherical mass distributions and that of a point mass[2]. Here we will do the same thing for charge distributions and we will do it by using the potential[3].

a) The situation is shown in figure 6. The sphere has a radius R and is *uniformly* charged with a charge density ρ. The distance from a contributing volume element dv is now $|\mathbf{r} - \mathbf{r}'|$, but this distance varies with θ so we need to need to relate the two. Use the law of cosines to show that $|\mathbf{r} - \mathbf{r}'|^2 = r^2 + r'^2 - 2rr' \cos\theta$ and use this to show that the potential can be written as

[2] Newton claimed he discovered the law of gravitation in his early years, but held back the publication for a long time because he could not prove the equivalence between spherical and point mass distributions. He first had to invent the calculus.

[3] Proving it by direct integration of the field is challenging, but absolutely doable.

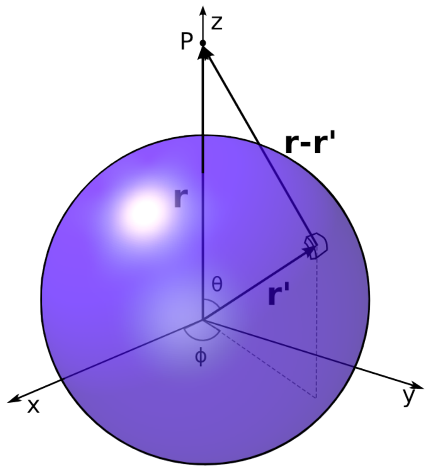

Figure 6: A sphere of uniform charge density.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho r'^2 \sin\theta d\phi d\theta dr'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}},$$

where this is evaluated as a triple integral. What are the limits of the integral?

Solution: In this problem it's important to keep in mind that $\mathbf{r}$ is the point we're trying to find the potential at, while $\mathbf{r}'$ is the point where the *source charge* dq' is located. When we are summing up (integrating) the contributions from all source charges dq' we integrating over r' and not r because r' is varying due to the different locations of the different $dq's$. We are trying to find

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq'}{|\mathbf{r} - \mathbf{r}'|}.$$

Using the law of cosines, or just using the dot product

$$\begin{aligned} (\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}') &= |(\mathbf{r} - \mathbf{r}')|^2 && (12) \\ &= \mathbf{r}\cdot\mathbf{r} + \mathbf{r}'\cdot\mathbf{r}' - 2\mathbf{r}\cdot\mathbf{r}' && (13) \\ &= r^2 + r'^2 - 2rr'\cos\theta. && (14) \end{aligned}$$

Now we are integrating over a volume, so $dq' = \rho(\mathbf{r}')dv' = \rho r'^2 \sin\theta d\theta d\phi dr'$ where $\rho(\mathbf{r}') = \rho$ is a constant because the sphere is uniformly charged. Thus

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq'}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho r'^2 \sin\theta d\theta d\phi dr'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}.$$

The limits of integration are $0 < \theta < \pi$, $0 < \phi < 2\pi$ and $0 < r' < R$.

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b) Evaluate the integral and find the potential *outside* the sphere. Show that it can be written as

$$V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}.$$

Hint: A simple substitution might be good here. Hint: $\sqrt{r^2 + r'^2 - 2rr'} = \sqrt{(r-r')^2} = |r - r'|$

Solution:

$$V(\mathbf{r}) = \frac{\rho}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^R \int_0^\pi \frac{r'^2 \sin\theta d\theta dr'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$$\begin{aligned}
\int_0^\pi \frac{r'^2 \sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} d\theta &= \frac{r'^2}{2rr'}\int_{r^2+r'^2-2rr'}^{r^2+r'^2+2rr'} u^{-1/2} du && (15)\\
&= \frac{r'}{r}\left(\sqrt{r^2 + r'^2 + 2rr'} - \sqrt{r^2 + r'^2 - 2rr'}\right) && (16)\\
&= \frac{r'}{r}\left(\sqrt{(r+r')^2} - \sqrt{(r-r')^2}\right) && (17)\\
&= \frac{r'}{r}\left(r + r' - |r - r'|\right) && (18)\\
&= \begin{cases} \frac{2r'^2}{r} & r > r' \\ 2r' & r < r' \end{cases} && (19)
\end{aligned}$$

where we have used the substitution $u = r^2 + r'^2 - 2rr'\cos\theta \Rightarrow du = 2rr'\sin\theta$. As long as we're outside the sphere r will always be greater than r', but in (d) we'll find the potential inside the sphere and then this is not the case. Thus

$$V(\mathbf{r}) = \frac{2\pi\rho}{4\pi\varepsilon_0}\left[\int_0^R \frac{2r'^2}{r} dr'\right] = \frac{(4/3\pi\rho R^3)}{4\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

where we have used that $Q = 4/3\pi R^3\rho$.

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c) Find the electric field from the sphere and compare it with the electric field from a point charge.

Solution:

$$\nabla V = \frac{Q}{4\pi\varepsilon_0}\nabla\frac{1}{r} = -\frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2} = -\mathbf{E}$$

such that

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}},$$

which is exactly the same as the field from a point charge.

Answer:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$$

d) Go a few steps back in your derivation above and find out what's different when we want to find the potential *inside* the sphere. Show that the potential here can be written as

$$V(\mathbf{r}) = \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

and find the electric field here. Is the field continuous in $r = R$?

Solution: When we're inside the sphere, $r < R$ and when integrating the contributions to the potential from all charges dq' r' can now be both greater and less than r. Thus

$$\begin{aligned} V(\mathbf{r}) &= \frac{2\pi\rho}{4\pi\varepsilon_0}\left[\int_0^r \frac{2r'^2}{r}dr' + \int_r^R 2r'dr'\right] && (20)\\ &= \frac{4\pi\rho}{4\pi\varepsilon_0}\left(\frac{r^3}{3r} + \frac{R^2 - r^2}{2}\right) && (21)\\ &= \frac{1}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) && (22) \end{aligned}$$

$$\boldsymbol{\nabla} V = \frac{Q}{4\pi\varepsilon_0 R}\boldsymbol{\nabla}\left(3 - \frac{r^2}{R^2}\right) = -\frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{\mathbf{r}} = -\mathbf{E}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}}$$

and in $r = R$ this gives

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\hat{\mathbf{r}}$$

so the electric field is continuous in $r = R$.

Answer:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}}$$