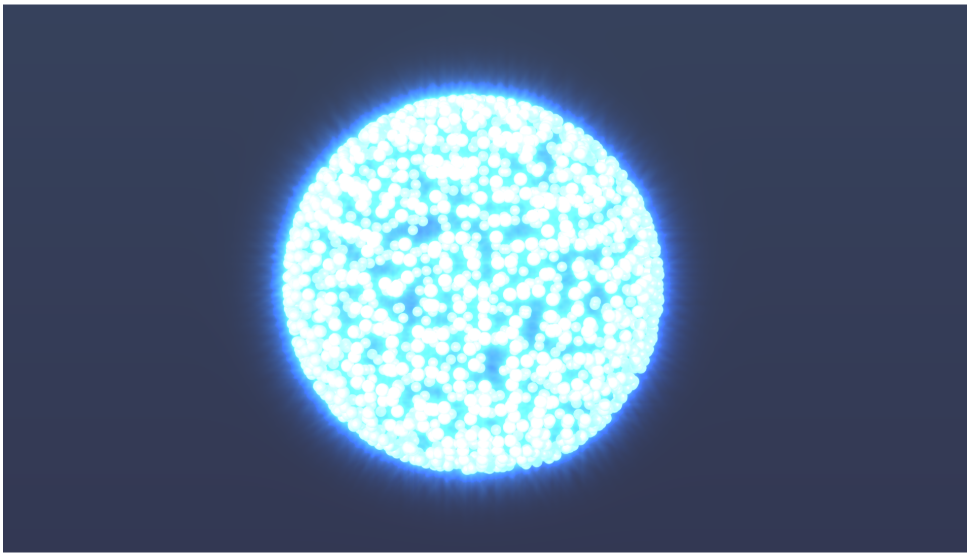
Week 4 - Oblig teaser, Gauss and marbles of electrons



> Further, the dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated.
>
> Carl Friedrich Gauss

In this problem set we're starting out with a teaser for the oblig that will be published later this week. The exercises refer to Jørgen Midtbø and Jørgen Trømborg's note on the Jacobi method that should have been published together with this problem set. Read that first and work through the first exercise in this problem set. The other exercises are unrelated to the note on the Jacobi method.

Exercise 4.1: Electrostatics with partial differential equations

a) Copy or rewrite the code from Section 2.2 in the note on the Jacobi method. Verify that it works and that you can produce something similar to Figure 1 in the same note.

b) Find the electric field **E**. If you use the NumPy `gradient()` function, remember that the step length h is part of the numerical derivative, and cannot be omitted.

c) Verify that Gauss's law is satisfied by calculating the flux of **E** out of the domain and comparing this to the charge q within.

d) What does the boundary condition $V = 0$ represent physically, i.e. what material(s) could you use to build something with this property?

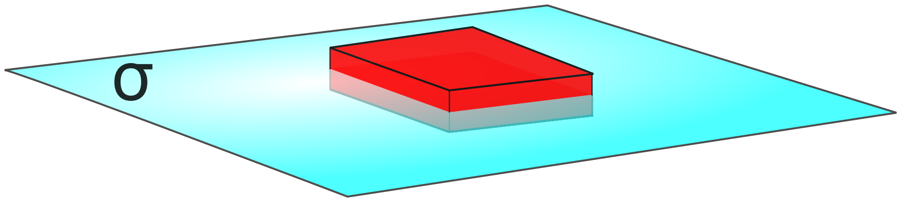


Figure 1: For an infinite plane the Gaussian surface might be a box. E will be constant along the top and bottom and the sides won't contribute to the integral since the electric field is perpendicular to them.

Gauss' law

Earlier we have looked at two methods of finding the electric field. Direct integration and by the electric potential. Finding the potential is usually easier than direct integration, and it can always be used, but in situations with great symmetry there exists an even easier method. The idea is to exploit *Gauss' law*. It states that

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \mathbf{da} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{\mathcal{V}} \rho d\tau,$$

or equivalently written in differential form $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$, where Q is the total charge enclosed any the surface S and ρ is the charge density which is generally a function of position. $\rho = \rho(r)$. Gauss law is a fundamental statement about electric fields that always holds, but it can be used to find the field itself. The key is to construct a surface $\mathcal{S}$, if such a surface exists, on which E is constant and points in the direction normal to the surface. Then

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \mathbf{da} = E \oint da = EA$$

and since the left side is known from the charge distribution we deduce $\mathbf{E}$. After some practice you'll be able to deduce some fields really fast using Gauss' law, so instead of remembering them you can deduce the fields quickly. Let's deduce some easy fields first

Exercise 4.2: Calculating the field

In the previous problem set we found the field from a finite plane and wire and took the limit as these geometrical objects got infinitely large. Of course no objects are really infinite, but the fields derived from such assumptions give good approximations to fields close to large objects.

a) By using Gauss' law, show that the fields from an infinitely long wire is

$$\mathbf{E} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r} \hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is now a cylindrical coordinate unit vector pointing away from the axis of symmetry.

Solution: The key is to exploit the symmetry of the problem. For an infinite wire the symmetry of the problem implies that the field has to be pointing radially away from the axis of symmetry and it has to be of equal strength at equal distances from this axis. So if we want a surface for which the field is equal in strength and is normal to the surface at all points a cylinder will do the job. We contruct a coaxial cylinder $\mathcal{S}$ with length L and radius r and consider

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A} = \int_{top+bottom} \mathbf{E} \cdot d\mathbf{A} + \int_{sides} \mathbf{E} \cdot d\mathbf{A} = 0 + E \int dA = E(2\pi r L)$$

such that by Gauss' law

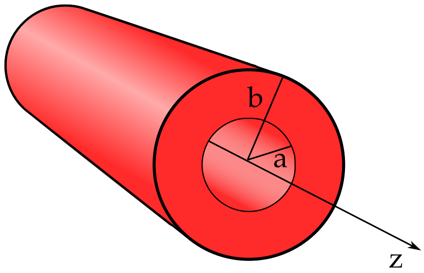


Figure 2: Cylinder with charge density ρ.

$$E2\pi rL = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$
$$\Rightarrow \mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$$

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b) Again using Gauss' law, show that the field from an infinite plane parallell with the xy-plane is

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}.$$

Solution: Construct a box like in figure **??**. Because the field is normal to the plane the flux integral get no contributions from the sides and therefore

$$\oint_S \mathbf{E}\cdot d\mathbf{A} = EA + EA = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

and thus

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}$$

where A is the area of the top and bottom of the box and $\hat{\mathbf{n}}$ is normal to the infinite plane.

Exercise 4.3: Non-uniform charge densities

In all the examples we've studied ρ has been uniform. But this need not be the case. In this example we'll study a long coaxial cable carrying a charge density $\rho(r) = kr$ in the region $a < r < b$ and $\rho = 0$ everywhere else, where r is the distance from the axis of symmetry and k is a constant. The situation is shown in figure 2.

a) Use Gauss' law to find the electric field inside for all r and plot E as a function of r.

Solution:

Construct cylinder with radius r. For $0 < r < a$ the enclosed charge is zero, so therefore the field here is also zero. For $a \leq r \leq b$

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A} = E2\pi r L = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int \rho(r) dv$$

$$Q_{enc} = \int \rho(r) dv = \int_0^L \int_0^{2\pi} \int_0^r kr^2 dr d\phi dz = 2\pi L \int_a^r kr^2 dr = \frac{2}{3} L\pi k \left(r^3 - a^3\right)$$

$$\Rightarrow \mathbf{E} = \frac{k\left(r^3 - a^3\right)}{3\varepsilon_0 r} \hat{r}.$$

For $r > b$ we essentially get the same derivation, but now

$$\mathbf{E} = \frac{k\left(b^3 - a^3\right)}{3\varepsilon_0 r} \hat{r}.$$

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b) Calculate the divergence of $\mathbf{E}$ and verify that $\rho(r) = \varepsilon_0 \nabla \cdot \mathbf{E}$.

Solution: Rewrite $\mathbf{E}$ as

$$\mathbf{E} = \frac{k}{3\varepsilon_0} \left(r^2 - \frac{a^3}{r}\right) \hat{\mathbf{r}}$$

such that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{k}{3\varepsilon_0} \left(\boldsymbol{\nabla} \cdot r^2 \hat{\mathbf{r}} - a^3 \boldsymbol{\nabla} \cdot \frac{\hat{\mathbf{r}}}{r}\right).$$

Now by either transforming this expression to cartesian coordinates or by using the divergence in cylindrical coordinates (search for 'del in cylindrical coordinates' on wikipedia) one can show that

$$\boldsymbol{\nabla} \cdot r^2 \hat{\mathbf{r}} = 3r$$

and

$$\boldsymbol{\nabla} \cdot \frac{\hat{\mathbf{r}}}{r} = 0.$$

Therefore

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{k}{3\varepsilon_0} 3r = \frac{\rho(r)}{\varepsilon_0}.$$

Exercise 4.4: The electron marble

We want to construct a marble solely out of electrons. Let's say we have the same glass marble as before, with $R = 1\,\mathrm{cm}$ in radius and weighing $m = 10\,\mathrm{g}$. Consider that we remove all the protons and are left with only the electrons. Last week we found that the electrons in this marble have a total charge of $Q = 48\,180\,\mathrm{C}$.

a) Let us model the marble as a sphere with uniform charge density ρ. Find the electric field inside and outside the sphere.

Hint: When you're inside the sphere, the total charge enclosed is not Q.

Note: This is the exact same problem as we had in the last exercise in the previous problem set. Do you find it easier to calculate the electric field with this method?

Answer: For $r < R$:

$$E = \frac{\rho}{3\varepsilon_0} r$$

For $r > R$:

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Solution: We use Gauss' law to determine the electric field for $r < R$ and $r > R$. Since the charges are placed uniformly over the volume, we can use a constant ρ as the charge density. For $r < R$ we have

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

which gives

$$\begin{aligned} E4\pi r^2 &= \frac{1}{\varepsilon_0} \int_V \rho dV \\ E4\pi r^2 &= \frac{\rho}{\varepsilon_0} \frac{4}{3} \pi r^3 \\ E &= \frac{\rho}{3\varepsilon_0} r \end{aligned}$$

Furthermore, for $r > R$ we get

$$\begin{aligned} E4\pi r^2 &= \frac{Q}{\varepsilon_0} \\ E &= \frac{Q}{4\pi\varepsilon_0 r^2} \end{aligned}$$

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b) To construct this marble, we assemble the electrons slowly one by one, until the charge is Q, and placed uniformly over the marble's volume. Due to the conservation of energy, the work done will be stored as energy in the the electric field. Therefore, you might want to use the energy density of the electric field,

$$u_E = \frac{1}{2} \varepsilon_0 E^2$$

to find the total energy. Find the total required work.

Hint: See chapter 2.4 in «Generell Fysikk».

Answer:

$$\frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}$$

Solution: The work needed is equal to the total energy stored in the electric field after the charges are assembled.

Now we may use that the energy density in the electric field is given as

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

We can integrate over all space where we have an electric field. To do this, we make up a small infinitesimal volume element $dV = 4\pi r^2 dr$ and see that this results in an infinitesimal energy element

$$dU = u_E dV = \frac{1}{2}\varepsilon_0 E^2 4\pi r^2 dr$$

For the field where $r < R$, we get

$$\begin{aligned} U_1 &= \int_0^R \frac{1}{2}\varepsilon_0 (\frac{\rho}{3\varepsilon_0} r)^2 4\pi r^2 dr \\ &= \int_0^R \frac{4\pi\rho^2 r^4}{2\cdot 3^2\varepsilon_0} dr \\ &= \frac{4\pi\rho^2 R^5}{5\cdot 2\cdot 3^2\varepsilon_0} \end{aligned}$$

Since the charge density is uniform, we have that the total charge must be the density times the volume, $Q = \rho 4/3\pi r^3$, so that

$$\rho = \frac{3}{4}\frac{Q}{\pi R^3}$$

Inserting for ρ we get the energy as

$$U_1 = \frac{Q^2}{5\cdot 2\cdot 4\pi\varepsilon_0 R}$$

For $r > R$ we get

$$U_2 = \int_R^\infty \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 r^2} dr = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R}$$

In total, this is

$$U = U_1 + U_2 = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}$$

This can also be solved by using

$$U = \frac{\rho}{2}\int_0^R 4\pi r^2 dr V(r)$$

or by modeling the total charge by bringing shells with charge $\rho 4\pi r^2 dr$ from $r \to \infty$ for a sphere of radius r and then integrate r from 0 to R.

By insertion of $Q = 48\,180\,\mathrm{C}$, we find

$$U = 2.6\cdot 10^{16}\,\mathrm{J}$$

c) Impressed as you are by the numbers, figure out how many trucks, all with an engine power of 745 kW (or 1000 HP, if you prefer) you'd need to push all this charge together. We're in no hurry, so let's say they'll use a day. [1]

Answer: 403928 trucks

Solution: The total energy produced by one truck would be

$$U_T = 745000\,\frac{\mathrm{J}}{\mathrm{s}} \cdot 24 \cdot 60 \cdot 60\,\mathrm{s} = 6.4368 \cdot 10^{10}\,\mathrm{J}$$

So the total number of trucks becomes

$$n = \frac{U}{U_T} = \frac{2.6 \cdot 10^{16}\,\mathrm{J}}{6.4368 \cdot 10^{10}\,\mathrm{J}} \approx 403928$$

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d) If we were actually able to assemble such a marble of electrons, the electrons would obviously want to immediately repel each other. The number of electrons where found last week to be $n_e = 3.011 \cdot 10^{23}$. If we assume that all the electrons are thrown out into space with an equal velocity for each electron, what would the speed of each electron be? (Is your answer reasonable or should we've considered the effects of relativity?)

Answer: $1.87 \cdot 10^{23}$ m/s. No it is not reasonable, and relativity should be taken into account.

Solution: The speed would be determined by the potential energy going over to kinetic energy. If each electron gets the same amount of the potential energy, each would gain

$$U_e = \frac{U}{n_e} = \frac{2.6 \cdot 10^{1}6\,\mathrm{J}}{3.011 \cdot 10^{23}} = 8.63 \cdot 10^{-8}\,\mathrm{J}$$

If all this potential energy was to be turned into kinetic energy, the velocity of each electron would be given by $U_e = 1/2 m_e v_e^2$.

$$v_e = \sqrt{\frac{2U_e}{m_e}} = 1.87 \cdot 10^{23}\,\mathrm{m/s}$$

which obviously is out of proportions compared to the speed of light. This means that the problem should be solved by proper use of relativity.

Exercise 4.5: Sensors

Since a conductor is an equipotential, we can unambiguously talk about the potential difference between them. Furthermore V is proportional to Q, so if we stick $+Q$ on one and $-Q$ on the other conductor we

[1] Don't worry about that the trucks would crash if the radius is 1 cm. This is a thought experiment, we're not really planning to do this.

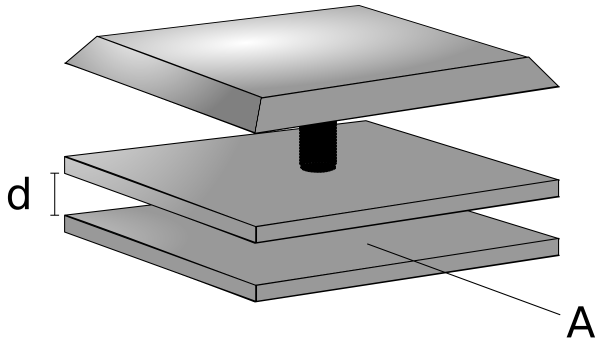

Figure 3: A keyboard application of a plate capacitor.

can define a useful quantity called the *capacitance* C of the arrangement by $C = Q/V$, where V here is the potential *difference* between the two conductors.

A parallel-plate capacitor is constructed by bringing two conducting plates with charge $\pm Q$ and of area A a distance d apart. As long as d is small in comparison to A and we stay away from the edges, we can model the field as the sum of the field from a pair of infinite plates[2].

a) Find the field everywhere for the plate capacitor described above and show that the capacitance can be written as

$$C = \frac{A\varepsilon_0}{d}\,.$$

Solution:

$$C = \frac{Q}{V}$$

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

(between the plates)

$$V = Ed = \frac{\sigma d}{\varepsilon_0} = \frac{Qd}{\varepsilon_0 A}$$

(homogeneous field)

$$\Rightarrow C = \frac{Q}{V} = \frac{A\varepsilon_0}{d}$$

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b) What is the force experienced by each of the plates? *Hint: Remember that a body cannot exert a force on itself.*

Solution: The upper plate can only exert a force on the lower plate and vise verca. The field from the plate with the positive charge Q is $\frac{Q}{2A\varepsilon_0}$ such that the magnitude force on the other plate is $F = \frac{Q^2}{2A\varepsilon_0}$.

[2] We've found this field before in exercise 3.3.c.

Answer:

$$F = \frac{Q^2}{2A\varepsilon_0}$$

c) One application of a capacitor is to use it as a sensor. This is used in some computers. Consider the button in figure 3 with $A = 50\,\mathrm{mm}^2$ and $d = 0.6\,\mathrm{mm}$. When you push a button, the distance d decreases and the capacitance goes up. When the capacitance get bigger than $C' = 0.250\,\mathrm{pF}$ the keystroke is registered. Find the distance we need to press the button down in order for the keystroke to be registered.

Solution: So the keystroke is registered when the capacitance is bigger than some minimal capacitance C'. Thus

$$C = \frac{A\varepsilon_0}{d} \geq C' \Leftrightarrow d \leq \frac{A\varepsilon_0}{C'}$$

Answer:

$$d \leq \frac{A\varepsilon_0}{C'}$$