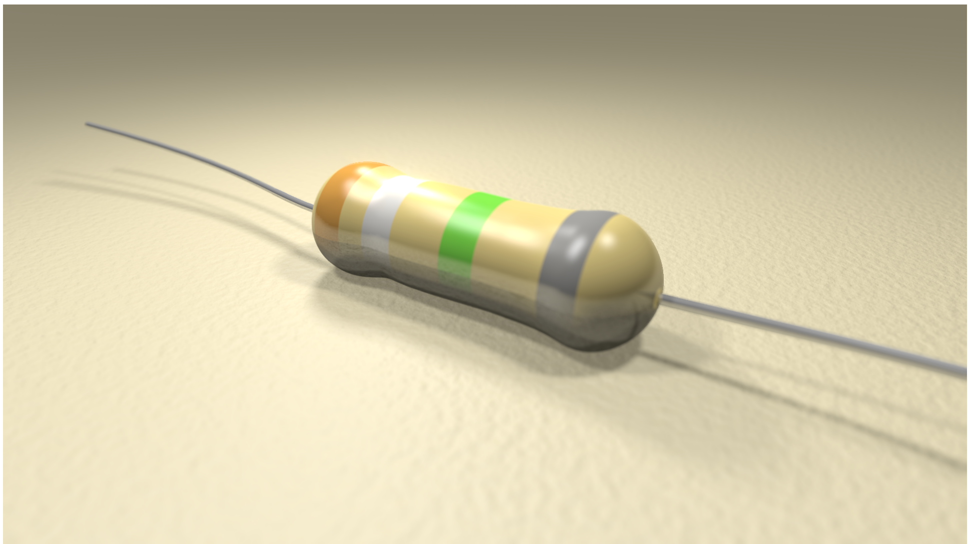
Week 5 - Conductors, capacitors and circuits



Each metal has a certain power, which is different from metal to metal, of setting the electric fluid in motion.

Count Alessandro Giuseppe Antonio Anastasio Volta

Exercise 5.1: Conductor properties

(i) $\mathbf{E} = 0$ inside a conductor.
(ii) $\rho = 0$ inside a conductor.
(iii) Any net charge resides on the surface. $\sigma \geq 0$
(iv) A conductor is an equipotential.
(v) $\mathbf{E}$ is perpendicular to the surface just outside the conductor.

If you look around, you probably have conductors all around you. They are invaluable in our technological society. In this exercise we'll study their *electrostatic* properties by studying hollow spherical conductor.

a) Consider the neutral spherical conductor in figure 1. We place a charge $-q$ inside a spherical cavity of radius a. Find the surface charge σ_a and σ_b.

Solution:

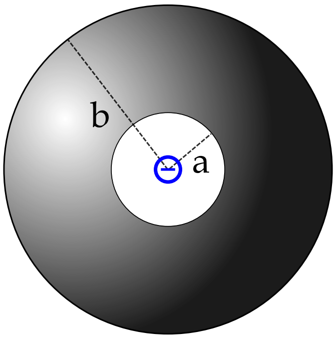


Figure 1: A hollow conductor with $-q$ inside.

The charge will arrange itself in such a way to cancel of the field inside the conductor. To cancel of the $-q$ charge inside q units of positive charge have to come to the boundary of radius a thus $\sigma_a = \frac{q}{4\pi a^2}$.

These charges had to come from somewhere. $\rho = 0$ inside the conductor which means they came from the surface leaving a negative surface charge density of $\sigma_b = -\frac{q}{4\pi b^2}$ behind.

Answer: $\sigma_a = \frac{q}{4\pi a^2}$ $\sigma_b = -\frac{q}{4\pi b^2}$

b) What is the electric field inside the cavity, as well as inside and outside the conductor?

Solution:

Inside the cavity

$$E = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2},$$

inside the conductor

$$E = 0,$$

while outside

$$E = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2}.$$

So the charge $-q$ communicates its existence to the outside world.

Answer:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2}$$

inside the cavity and outside the conductor. Inside the conductor $\mathbf{E} = 0$.

c) What is the force on the charge inside?

Solution:

The uniform charge distibutions make the net force equal to zero

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Answer: Zero.

d) If we remove the charge inside and expose the conductor to some huge electric fields from the outside. Do you know whether or not it would be safe to sit inside the cavity? Are there any fields inside? Explain.

Exercise 5.2: Coaxial cylinders

Consider the coaxial conducting cylinders in figure 2 of radii a and b and length L.

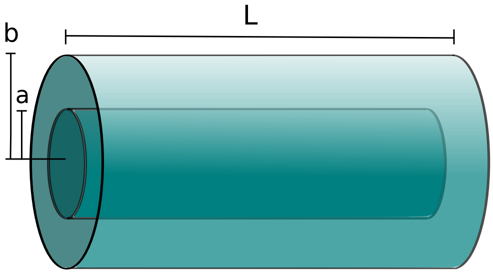

Figure 2: Capacitor consisting of two coaxial cylinders.

a) Given that the inner cylinder has charge $-Q$ and the outer has Q. Ignore the fringing fields and use Gauss law to find the electric field between the cylinders.

Solution: Construct an imaginary cylinder of length L in between the two cylinders. Because the positive charge is on the outer cylinder the field will anti parallel to $d\mathbf{A}$ at every point which gives

$$\oint \mathbf{E} \cdot d\mathbf{A} = -E2\pi Lr = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \mathbf{E} = -\frac{Q}{2\pi\varepsilon_0 Lr}\hat{\mathbf{r}}$$

Answer:

$$\mathbf{E} = -\frac{Q}{2\pi\varepsilon_0 Lr}\hat{\mathbf{r}}$$

b) Show that the capacitance is given by

$$C = \frac{2\pi\varepsilon_0 L}{\ln \frac{b}{a}}$$

Solution: The capacitace is defined as $C = Q/V$ so we first have to find the potential difference between the two cylinders V.

$$V = V_b - V_a = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{r} = \frac{Q}{2\pi\varepsilon_0 L} \int \frac{1}{r} dr = \frac{Q}{2\pi\varepsilon_0 L} \ln \frac{b}{a}$$

we then get

$$C = \frac{Q}{\frac{Q}{2\pi\varepsilon_0 L} \ln \frac{b}{a}} = \frac{2\pi\varepsilon_0 L}{\ln \frac{b}{a}}.$$

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c) Find the energy stored in the electric field between the cylinders given that $a = 1\,\text{mm}$, $b = 2\,\text{mm}$ and $Q = 0.2\,\text{nC}$.

Solution: In general the energy between different objects i with charge Q_i can be written $U = \frac{1}{2}\sum_i Q_i V_i$. For a capacitor with opposite charge on the two conductors this implies that $U = \frac{1}{2} Q \Delta V$, and $C = Q/V$ so that

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Answer:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

d) Suppose now that we place this capacitor partially in water with $\kappa = 80$ and that the total charge on the capacitor was unaffected. See figure 3. Find an expression for V in the two regions.

Hint: The polarized water will change the surface charge on the part of the capacitor immersed

Answer:

$$V_{air} = \frac{\sigma_{air} a}{\varepsilon_0} \ln \frac{b}{a}$$

$$V_{water} = \frac{\sigma_{water} a}{\kappa\varepsilon_0} \ln \frac{b}{a}$$

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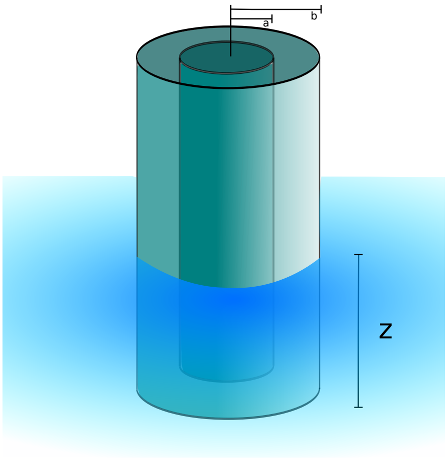

Figure 3: Cylinder capacitor used as a water height sensor.

Solution:

$$E_{air} = -\frac{\sigma_{air} a}{\varepsilon_0 r}$$

while

$$E_{water} = -\frac{\sigma_{water} a}{\kappa \varepsilon_0 r}$$

which gives

$$V_{air} = \frac{\sigma_{air} a}{\varepsilon_0} \ln \frac{b}{a}$$

and

$$V_{water} = \frac{\sigma_{water} a}{\kappa \varepsilon_0} \ln \frac{b}{a}$$

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e) Now we know that a conductor is an equipotential. Can you use this to find a relation between the surface charge in the two regions? Use this to find the capacitance $C = C(z)$.

Answer:

$$\sigma_{water} = \kappa \sigma_{air}$$

$$C = \frac{Q}{V} = 2\pi\varepsilon_0 \frac{(z(\kappa - 1) + L)}{\ln \frac{b}{a}}.$$

Solution:

A conductor is an equipotential $V_{air} = V_{water}$ which means that

$$\sigma_{water} = \kappa \sigma_{air}.$$

Now the total charge on the capacitor is

$$Q = \sigma_{water} 2\pi a z + \sigma_{air} 2\pi a (L - z) = 2\pi a \sigma \left(z(\kappa - 1) + L \right)$$

such that

$$C = \frac{Q}{V} = 2\pi\varepsilon_0 \frac{(z(\kappa - 1) + L)}{\ln \frac{b}{a}}.$$

Exercise 5.3: Electric circuits

Consider the circuit shown in figure 4 where $R_1 = 10\,\Omega$, $R_2 = 4\,\Omega$ and $R_3 = 8\,\Omega$. The battery has a voltage of $\varepsilon = 3\,\text{V}$.

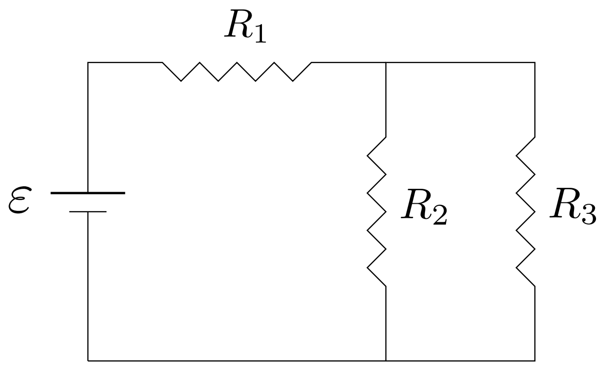

Figure 4: One resistor in series and two in parallel.

a) What is the current through the battery?

Solution: Naming the current through resistor R_i for I_i, where $i = 1, 2, 3$. Using Kirchoff's voltage law for the rightmost loop:

$$\varepsilon - I_1 R_1 - I_2 R_2 = 0$$

For the leftmost:

$$I_3 R_3 - I_2 R_2 = 0$$

The current law:

$$I_1 = I_2 + I_3$$

Inserting this into the leftmost:

$$\varepsilon - I_1 R_1 - R_2 I_1 + I_3 R_2 = 0$$

solving for I_3:

$$I_3 = \frac{I_1 R_1 + R_2 I_1 - \varepsilon}{R_2}$$

Inserting I_3 into the rightmost gives

$$I_3R_3 - (I_1 - I_3)R_2 = 0$$
$$R_3\frac{I_1R_1 + R_2I_1 - \varepsilon}{R_2} - I_1R_2 + R_2\frac{I_1R_1 + R_2I_1 - \varepsilon}{R_2} = 0$$
$$I_1 = \frac{\varepsilon(R_3 + R_2)}{R_1R_2 + R_1R_3 + R_3R_2} = \frac{\varepsilon}{R_1 + \frac{R_3R_2}{R_2+R_3}}$$

where I've omitted the intermediary steps.

You could also use the useful shorthand $R_{\text{tot}} = R_1 + R_2||R_3 = R_1 + \frac{R_2R_3}{R_2+R_3}$ and use $I_1 = \varepsilon/R_{\text{tot}}$ to find the current. This is however not part of this course, so you're safest off with Kirchoff if you haven't seen this before.

Answer:

$$I_{\text{tot}} = I_1 = 0.237\,\text{A}$$

b) What is the total resistance in the circuit?

Solution:

$$R_{\text{tot}} = \frac{\varepsilon}{I_{\text{tot}}} = R_1 + \frac{R_2R_3}{R_2 + R_3}$$

Answer:

$$R_{\text{tot}} = 12.7\,\Omega$$

c) What are the currents through R_2 and R_3?

Solution:

$$V_{R_2} = V_{R_3} = \varepsilon - R_1I_{\text{tot}}$$
$$I_{R_2} = \frac{V_{R_2}}{R_2} = \frac{\varepsilon - R_1I_{\text{tot}}}{R_2}$$
$$I_{R_3} = \frac{V_{R_3}}{R_3} = \frac{\varepsilon - R_1I_{\text{tot}}}{R_3}$$

Answer:

$$I_{R_2} = 0.15$$
$$I_{R_3} = 0.075$$

d) What is the power generated of the battery?

Solution:

$$P = VI$$

Answer:

$$P = 3\,\mathrm{V} \cdot 0.24\,\mathrm{A} = 0.7\,\mathrm{W}$$

e) What is the total power consumed in the resistors?

Solution:

$$P = I^2 R_{\mathrm{tot}}$$

Answer:

$$P = (0.24\,\mathrm{A})^2 \cdot 12.7\,\Omega = 0.7\,\mathrm{W}$$