Fys2160 – 2013 – Oblig 2

Equilibrium fluctuations in a spin system

In this project we will characterize micro- and macro-states in a spin system and address fluctuations between two thermally coupled spin systems. You will learn how to calculate fluctuations analytically and numerically and how to relate the multiplicities to entropy and temperature of the system.

In a paramagnetic system with binary spins, each particle can be in two possible states, $S = +1$ or $S = -1$. We call such a "particle" a spin – since it is only the spin we are interested in. The energy of a single spin depends on the orientation of the spin relative to an external magnetic field B. If the spin is parallell (or antiparallell) to the external field, the energy is $E = -S\mu B$, where $S = +1$ corresponds to a spin parallell to the field and $S = -1$ corresponds to a spin antiparallell to the field.

The simplest system we can consider consists of N independent spins, S_i. (Independent here means that the spins do not interact, just like for an ideal gas or an Einstein crystal).

(a) What is the number of microstates of the N-spin system?

We introduce S_+ as the number of spins with value $+1$ and S_- as the number of spins with value -1. We call $2s = S_+ - S_-$ the spin excess or the net spin.

(b) Express the total energy using the net spin, $2s$.

(c) Generate 10000 microstates for a $N = 50$ system randomly – assuming that all microstates are equally likely – and plot the energy as the number of the trial. Plot a histogram of the energies using for example `hist` in matlab.

(d) Show that the multiplicity of a state with S_+ spins pointing up is

$$\Omega(N, S_+) = \frac{N!}{S_+!S_-!} \; . \tag{7}$$

(e) Show that the multiplicity can be written as a function of the net spin, s, on the following form:

$$\Omega(N, s) = \frac{N!}{\left(\frac{1}{2}N + s\right)! \left(\frac{1}{2}N - s\right)!} \; , \tag{8}$$

Since the multiplicity is a very large number when N is large, you may in the following want to work using the logarithm of the multiplicity.

(f) (This problem requires some algebra – you may skip it without loss of continuity). Show that the multiplicity can be written as

$$\Omega(N, s) = \Omega(N, 0) \exp(-2s^2/N) \; , \tag{9}$$

using Stirling's formula:

$$N! \simeq (2\pi N)^{1/2} N^N e^{-N} \; , \tag{10}$$

(g) Compare this analytical result with the histogram you generated of the microstates.

We will now study a system consisting of two spin systems each with N_1 and N_2 spins. The net spin in the whole system is $2s = N_+ - N_-$ and $2s_1$ and $2s_2$ in each of the subsystems. The total energy (and therefore the total net spin) is conserved, but the two subsystems may exchange energy.

You can assume that $N_1 = N_2 = 10^{22}$ and that the multiplicities therefore have the sharp, Gaussian form:

$$\Omega_1(N_1, s_1) = \Omega_1(N_1, 0)\exp(-2s_1/N_1) \; , \tag{11}$$

and similarly for s_2.

(h) What are the possible values of s_1 (or $2s_1$ if your prefer). What are the possible energies U_1 of system 1? We call each such a state (with a given energy U_1 in system 1) a macrostate of the coupled system.

(i) Show that the multiplicity of the macrostate s_1, s_2 is

$$\Omega_1(N_1, 0)\Omega_2(N_2, 0)\exp\left(-\frac{2s_1^2}{N_1} - \frac{2s_2^2}{N_2}\right) \; . \tag{12}$$

(j) (This requires some algebra, but is fully possible). Show that in the most probable macrostate we have $\hat{s}_1$ and $\hat{s}_2$ where

$$\frac{\hat{s}_1}{N_1} = \frac{\hat{s}_2}{N_2} = \frac{s}{N} \; . \tag{13}$$

We will now return to a single system with N spins and S_+ spins parallell to the magnetic field.

(k) Find the entropy as a function of N and S_+.

(l) Find an expression for the temperature of the system.

(Hint: $\partial S/\partial U = (\partial S/\partial S_+)\,(\partial S_+/\partial U) = (\partial S/\partial S_+)\,(\partial s/\partial U)$.)

End of Oblig 2