Solutions to exercises week 35

FYS2160

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Compendium 4.1

a)

$P(\text{an ace of spades}) = \frac{1}{52}$

b)

$P(\text{a 2 of spades}) = \frac{1}{52}$

c)

$P(\text{a black card}) = \frac{26}{52} = \frac{1}{2}$

d)

$P(\text{a spade}) = \frac{13}{52} = \frac{1}{4}$

e)

$P(\text{not a spade}) = 1 - P(\text{a spade}) = 1 - \frac{13}{52} = \frac{1}{4}$

f)

$P(\text{two spades}) = \frac{13}{52} \cdot \frac{12}{51}$

g)

$P(\text{five cards, four of equal value}) = \frac{13 \cdot 48}{\binom{52}{5}} = 13 \cdot 48 \cdot \frac{5! \cdot 47!}{52!} \approx 2.4 \times 10^{-4}$.
A deck of cards has four suits, and each suit has 13 cards of different value. Then there are 13 ways of drawing four cards of equal value (not caring about in which order they are drawn), and for each of these 13 sets there are 48 possibilities for the fifth card. We divide this by the total number of ways of drawing 5 cards out of 52 (unordered), which is equal to the binomial coefficient $\binom{52}{5}$ ("52 choose 5").

Compendium 4.2

a)

$P(\text{throw one die, get a 1}) = \frac{1}{6}$

b)

$P(\text{throw one die, not get a 1}) = 1 - P(\text{throw one die, get a 1}) = 1 - \frac{1}{6} = \frac{5}{6}$

c)

$P(\text{throw two dice, get two 6s}) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

d)

$P(\text{throw two dice, get sum 3}) = \frac{2}{36} = \frac{1}{18}$
There are two ways of getting a sum of 3 when throwing two dice: 1+2 and 2+1; with 36 different outcomes in total.

e)

Sum j	Outcomes	# of outcomes	$P(j)$
2	1+1	1	$\frac{1}{36}$
3	1+2, 2+1	2	$\frac{2}{36}$
4	1+3, 2+2, 3+1	3	$\frac{3}{36}$
5	1+4, 3+2, 2+3, 4+1	4	$\frac{4}{36}$
6	1+5, 2+4, 3+3, 4+2, 5+1	5	$\frac{5}{36}$
7	1+6, 2+5, 3+4, 4+3, 5+2, 6+1	6	$\frac{6}{36}$
8	2+6, 3+5, 4+4, 5+3, 6+2	5	$\frac{5}{36}$
9	3+6, 4+5, 5+4, 6+3	4	$\frac{4}{36}$
10	4+6, 5+5, 6+4	3	$\frac{3}{36}$
11	5+6, 6+5	2	$\frac{2}{36}$
12	6+6	1	$\frac{1}{36}$
		sum = 36	

One can also find a general expression for the probability:
$P(\text{two dice, sum } j) = \frac{6-|7-j|}{36}$.

f)

$P(\text{throw two dice twice, get two 6s at least once})$
$= 1 - P(\text{throw two dice twice, not get two 6s in either throw}) = 1 - \frac{35}{36} \cdot \frac{35}{36}$.
There are 36 possible outcomes in total when throwing two dice. Out of these 36, only one outcome has both dice showing the value 6. Therefore, the probability of not getting two 6s when throwing two dice is $\frac{35}{36}$, and the probability of this happening in both throws is $\frac{35}{36} \cdot \frac{35}{36}$.

g)

$P(\text{four dice, at least two 6s}) = 1 - P(\text{four dice, less than two 6s})$

$= 1 - [P(\text{four dice, one 6}) + P(\text{four dice, no 6s})]$

$= 1 - [\text{Binomial}(k = 1, N = 4) + \text{Binomial}(k = 0, N = 4)]$

$= 1 - \left[\binom{4}{1}(\frac{1}{6})^1(\frac{5}{6})^3 + \binom{4}{0}(\frac{1}{6})^0(\frac{5}{6})^4\right]$

$= 1 - \left[4\frac{5^3}{6^4} + 1\frac{5^4}{6^4}\right] = 1 - \frac{125}{144} = \frac{19}{144}$

h)

$P(\text{two dice, both show same value})$

$= P(\text{first die, any value}) \cdot P(\text{second die, same as first value})$

$= 1 \cdot \frac{1}{6} = \frac{1}{6}.$

There are no restrictions on what the value on the first die should be, hence all six outcomes are valid, and the probability of getting any value is $6 \cdot \frac{1}{6} = 1$. Since there is only one outcome for the second die that gives the same value as the first die, the probability of this is $\frac{1}{6}$.

i)

$P(\text{two dice, odd sum})$

$= P(\text{first die, any value}) \cdot P(\text{second die, a value that gives odd sum})$

$= 1 \cdot \frac{1}{2} = \frac{1}{2}.$

No restrictions on the value on the first die. If the first value is an odd number, the second needs to be even to give an odd sum. There are then three out of six possible outcomes for the second die that give an odd sum. A similar argument applies if the first value is even; then there is a $\frac{1}{2}$ probability of getting an odd value on the second die. The probability of getting an even sum can be found by corresponding arguments, or by noting that $P(\text{two dice, even sum}) = 1 - P(\text{two dice, odd sum}) = \frac{1}{2}$.

j)

$P(\text{one die six times, at least one 6}) = 1 - P(\text{one die six times, no 6s})$
$= 1 - \left(\frac{5}{6}\right)^6 \approx 67\ \%$.

Compendium 4.3

a)

$P(\text{heads}) = p$ and $P(\text{tails}) = 1 - p$. The probability to get tails on the first $N - 1$ flips, and then a heads on the N'th flip is: $P(\text{N}) = (1 - p)^{N-1}p$.

b)

Need to show that

$$\sum_{N=1}^{\infty} P(N) = 1.$$

We will use the following relation for a geometric series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

when $|x| < 1$. The sum of $P(N)$ over all N can then be expressed as

$$\begin{aligned}\sum_{N=1}^{\infty} P(N) &= \sum_{N=1}^{\infty} (1-p)^{N-1}p \\ &= p \sum_{N=1}^{\infty} (1-p)^{N-1}.\end{aligned}$$

Substituting the summing index, $M = N - 1$, we recognize the expression for a geometric series, and get

$$\begin{aligned}\sum_{N=1}^{\infty} P(N) &= p \sum_{M=0}^{\infty} (1-p)^{M} \\ &= p\frac{1}{1-(1-p)} \\ &= p\frac{1}{p} = 1,\end{aligned}$$

which shows that $P(N)$ is normalized.

c)

Defining $q = 1 - p$, the average, or expectation value, of N, given the probability $P(N)$, is

$$\begin{aligned}\langle N\rangle &= \sum_{N=1}^{\infty} NP(N) = \sum_{N=1}^{\infty} Nq^{N-1}p \\ &= \sum_{N=1}^{\infty} N\frac{q^N}{q}p = \frac{p}{q}\sum_{N=1}^{\infty} Nq^N.\end{aligned}$$

We notice that summing from $N = 0$ doesn't add anything to the series, because $0 \cdot q^0 = 0$. We then recognize the sum given in the question, and use the corresponding relation

$$\sum_{N=0}^{\infty} Nq^N = \frac{q}{p^2}.$$

This results in

$$\langle N\rangle = \frac{p}{q}\sum_{N=0}^{\infty} Nq^N = \frac{p}{q}\frac{q}{p^2} = \frac{1}{p}.$$

d)

The expectation value of the return $r(N) = 2^N$ is

$$\begin{aligned}\langle r\rangle &= \sum_{N=1}^{\infty} r(N)P(N) = \sum_{N=1}^{\infty} 2^N q^{N-1}p \\ &= 2p\sum_{N=1}^{\infty} (2q)^{N-1}.\end{aligned}$$

Again substituting the summing index, $M = N - 1$, and recognizing the expression for a geometric series, we get

$$\begin{aligned}\langle r\rangle &= 2p\sum_{M=0}^{\infty} (2q)^M = 2p\frac{1}{1-2q} \\ &= \frac{2p}{1-2(1-p)} = \frac{2p}{2p-1} = \frac{1}{1-\frac{1}{2p}}.\end{aligned}$$

When $p \to 1/2$ from above, $\langle r\rangle \to \infty$. Hence, the casino would need an unfair coin, with probability $p > 1/2$, in order to avoid losing money on the game.

e)

If $p = 0.6$, the entry fee for the game needs to be at least

$$\langle r(p=0.6) \rangle = \frac{1}{1 - \frac{1}{2\cdot 0.6}} = 6 \text{ rubles}$$

in order to avoid losing money on the game in the long run.

Compendium 4.4

a)

```
M = 1E4; % number of trials
c1 = 0;  % counter for how many times person 1 has won
c2 = 0;  % counter for how many times person 2 has won

% Loop to simulate M=10000 coin-flip games.
% Draw random float, x, in the range [0,1) for person 1 and 2.
% Let 0 <= x < 0.5 represent tails, and 0.5 <= x < 1 represent heads
while c1+c2 < M         % continue flipping coins until 10000 games have been won
    x1 = rand;          % draw random number for person 1
    x2 = rand;          % draw random number for person 2
    if x1 >= 0.5        % check first if person 1 wins the game
        c1 = c1 + 1;    % add 1 to person 1's victory counter
        continue        % person 1 has won the game, start new game
    elseif x2 >= 0.5    % if person 1 did not get heads, check if person 2 did
```

```
        c2 = c2 + 1;   % add 1 to person 2's victory counter
    end
end
c1              % print number of times person 1 has won
N = c1 + c2     % print total number of wins (check that it's
   equal to M)
p1 = c1/N       % print estimated probability that person 1 wins
    the game
```

The probability that the person who starts flipping wins the game, is found to be approximately 0.67.

b)

Now, finding an analytical solution to the problem: Let p_1 be the probability that the person that starts flipping (person 1) wins the game, and assume the coin to be fair with probability 1/2 of showing heads and 1/2 of showing tails. Person 1 wins if he/she flips a head on the first flip, with probability 1/2. If person 1 instead gets a tail on the first flip, person 2 must then also flip a tail on the following flip, in order for person 1 to stilll have a chance to win. The probability of flipping tails two times in a row is 1/4. But then we are back in the initial situation, with person 1 having the same probability to win as before the first flip. We can express the probability for person 1 to win as

$$p_1 = \frac{1}{2} + \frac{1}{4}p_1 \quad \Rightarrow \quad p_1 = \frac{2}{3},$$

where we have solved the expression for p_1 and found that the person that starts flipping has a 2/3 probability of winning the game. This is consistent with the result of the numerical simulation in a).

Compendium 4.5

a)

We divide the range $[0,x]$ in n intervals of width Δx, giving $x = n\Delta x$ and $\Delta x = x/n$. We then get the following probabilities:

$$
\begin{aligned}
P(\text{scattered in the interval } \Delta x) &= p\Delta x, \\
P(\text{not scattered in the interval } \Delta x) &= 1 - p\Delta x, \\
P(\text{not scattered in the interval n}\Delta x) &= (1 - p\Delta x)^n = \left(1 - p\frac{x}{n}\right)^n \\
&= \left[1 + \left(\frac{-px}{n}\right)\right]^n \xrightarrow[n\to\infty]{} e^{-px},
\end{aligned}
$$

where we in the last step have used the relation

$$
\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.
$$

b)

$$
\begin{aligned}
dP(\text{scattered in } [x, x+dx]) &= P(\text{not scattered in } [0,x]) \cdot dP(\text{scattered in interval } dx) \\
&= e^{-px} p dx.
\end{aligned}
$$

c)

The expected distance traveled by an electron before it is scattered, is

$$
\langle x \rangle = \int_0^\infty x P(x) dP = \int_0^\infty x e^{-px} p dx = p \int_0^\infty x e^{-px} dx.
$$

By partial integration with $u = x$ and $v' = e^{-px}$, we get

$$
\langle x \rangle = - \left. x e^{-px} \right|_0^\infty + \int_0^\infty e^{-px} p dx = 0 + \int_0^\infty e^{-px} p dx.
$$

Substituting $w = -px$, we end up with

$$
\langle x \rangle = -\frac{1}{p} \int_0^{-\infty} e^w dw = \frac{1}{p} \left[e^w\right]_{-\infty}^0 = \frac{1}{p}.
$$

d)

The probability of the electron not being scattered before a distance x is reached, is

$$P(x) = e^{-px}.$$

Simply inserting the relation $x = vt$ in $P(x)$, we get the probability of an electron not being scattered before a time t,

$$P(t) = e^{-pvt} = e^{-t/\tau},$$

where $\tau \equiv 1/pv$.

e)

Similar to the calculation of the expected distance traveled before scattering, the expected time traveled before scattering is

$$\langle t \rangle = \int_0^\infty t e^{-pvt} pv dt = \frac{1}{pv} = \tau,$$

where τ is the mean-free-time between scattering.

Compendium 4.6 (not given as a weekly exercise)

a)

$$P(N, 1) = (1 - p)^{N-1} p.$$

b)

This is simply the probability mass function of the binomial distribution

$$P(N, n) = \binom{N}{n} p^n (1 - p)^{N-n}.$$

The average, or expected value, of n is

$$\begin{aligned}
\bar{n} = \langle n \rangle &= \sum_{n=0}^{\infty} nP(N,n) \\
&= \sum_{n=0}^{\infty} n \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \\
&= Np \sum_{n=0}^{\infty} n \frac{N!}{n!(N-1-n+1)!} p^{n-1} (1-p)^{N-1-n+1} \\
&= Np \sum_{n=1}^{\infty} \frac{(N-1)!}{(n-1)![(N-1)-(n-1)]!} p^{n-1} (1-p)^{(N-1)-(n-1)} \\
&= Np \sum_{n=1}^{\infty} \binom{N-1}{n-1} p^{n-1} (1-p)^{(N-1)-(n-1)}, \quad m = n-1,\, M = N-1 \\
&= Np \sum_{m=0}^{\infty} \binom{M}{m} p^m (1-p)^{M-m} \\
&= Np(p + (1-p))^m, \quad \text{where the binomial theorem has been used} \\
&= Np \cdot 1^m \\
&= Np.
\end{aligned}$$

c)

From b) we know that $\bar{n} = Np$, which gives $p = \bar{n}/N$. Inserting this in $P(N,n)$ gives

$$\begin{aligned}
P(N,n) &= \binom{N}{n} p^n (1-p)^{N-n} \\
&= \binom{N}{n} \frac{\bar{n}^n}{N^n} \left(1 - \frac{\bar{n}}{N}\right)^{N-n} \\
&= \frac{N!/(N-n)!}{N^n} \frac{\bar{n}^n}{N!} \left(1 - \frac{\bar{n}}{N}\right)^{N} \left(1 - \frac{\bar{n}}{N}\right)^{-n}.
\end{aligned}$$

In the limit where $N \to \infty$ and $\bar{n} = Np$ is held fixed, which means $p \ll 1$, we get

$$
\begin{aligned}
\lim_{N\to\infty} P(N,n) &= \lim_{N\to\infty} \underbrace{\frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-n+1}{N}}_{1} \frac{\bar{n}^n}{N!} \underbrace{\left(1-\frac{\bar{n}}{N}\right)^N}_{e^{-\bar{n}}} \underbrace{\left(1-\frac{\bar{n}}{N}\right)^{-n}}_{1} \\
&= \frac{\bar{n}^n}{N!} e^{-\bar{n}}.
\end{aligned}
$$

Compendium 4.9

a)

$P(\text{N flips, all heads}) = p^N$

b)

$P(\text{N flips, n heads, N} - \text{n tails}) = P(N,n) = \binom{N}{n} p^n (1-p)^{N-n}$

c)

From 4.5 b) we have

$$
\bar{n} = \langle n \rangle = \sum_{n=0}^{\infty} n P(N,n) = Np.
$$

To calculate the variance $\overline{(n-\bar{n})^2} = \overline{n^2} - \bar{n}^2$, we first need to find an expression for $\overline{n^2}$:

$$\begin{aligned}
\overline{n^2} = \langle n^2 \rangle &= \sum_{n=0}^{\infty} n^2 P(N,n) \\
&= \sum_{n=0}^{\infty} n^2 \binom{N}{n} p^n q^{N-n} \\
&= \sum_{n=0}^{\infty} (n^2 - n + n) \binom{N}{n} p^n q^{N-n} \\
&= \sum_{n=0}^{\infty} (n^2 - n) \binom{N}{n} p^n q^{N-n} + \sum_{n=0}^{\infty} n \binom{N}{n} p^n q^{N-n}.
\end{aligned}$$

Using that

$$p\frac{d}{dp}p^n = npp^{n-1} = np^n,$$

and

$$p^2\frac{d^2}{dp^2}p^n = p^2 n \frac{d}{dp}p^{n-1} = p^2 n(n-1)p^{n-2} = n(n-1)p^n = (n^2-n)p^n,$$

we get

$$\begin{aligned}
\overline{n^2} &= \sum_{n=0}^{\infty} (n^2 - n) \binom{N}{n} p^n q^{N-n} + \sum_{n=0}^{\infty} n \binom{N}{n} p^n q^{N-n} \\
&= p^2 \frac{d^2}{dp^2} \sum_{n=0}^{\infty} \binom{N}{n} p^n q^{N-n} + p\frac{d}{dp} \sum_{n=0}^{\infty} \binom{N}{n} p^n q^{N-n} \\
&= p^2 \frac{d^2}{dp^2} (p+q)^N + p\frac{d}{dp}(p+q)^N \\
&= Np^2 \frac{d}{dp}(p+q)^{N-1} + Np(p+q)^{N-1} \\
&= N(N-1)p^2(p+q)^{N-2} + Np(p+q)^{N-1} \\
&= N(N-1)p^2(p+1-p)^{N-2} + Np(p+1-p)^{N-1} \\
&= N(N-1)p^2 + Np \\
&= N^2p^2 - Np^2 + Np \\
&= N^2p^2 - Np(1-p).
\end{aligned}$$

Finally, we can calculate the variance

$$\overline{n^2} - \bar{n}^2 = (Np)^2 - [N^2p^2 - Np(1-p)] = Np(1-p) = Npq.$$

d)

$$\begin{aligned} P(\text{N squares, all conducting}) &= p^N, \\ P(\text{N squares, all but one conducting}) &= P(N, N-1) \\ &= \frac{N!}{(N-1)![N-(N-1)]!} p^{N-1}(1-p)^{N-(N-1)} \\ &= \frac{N!}{(N-1)!1!} p^{N-1}(1-p) \\ &= Np^{N-1}(1-p), \end{aligned}$$

where we don't care about which square is isolating.

e)

$$\begin{aligned} P(\text{one isolating column}) &= (1-p)(1-p) = (1-p)^2, \\ P(\text{one conducting column}) &= 1 - P(\text{one isolating column}) \\ &= 1 - (1-p)^2, \\ P(\text{N conducting columns}) &= [P(\text{one conducting column})]^N \\ &= [1-(1-p)^2]^N. \end{aligned}$$

f)

Total number of different configurations when there are $2N$ squares of which $N+M$ are conducting:

$$\Omega_{\text{tot}} = \binom{2N}{N+M} = \frac{(2N)!}{(N+M)![2N-(N+M)]!} = \frac{(2N)!}{(N+M)!(N-M)!}.$$

g)

For the double strip to be conducting, there needs to be at least one conducting square per column. To find all conducting arrangements, we first consider placing one conducting square in every column. With two squares in each column, there are 2^N ways we can construct a conducting configuration in this way. Since there are $N + M$ conducting squares in total, we have M conducting squares to distribute among the remaining N squares in the double strip. There are $\binom{N}{M}$ ways to arrange these for each of the 2^N conducting configurations. The total number of conducting configurations are then

$$\Omega_{\text{cond}} = 2^N \binom{N}{M} = 2^N \frac{N!}{M!(N-M)!}.$$

h)

$$\begin{aligned} P_{\text{cond}} = \frac{\Omega_{\text{cond}}}{\Omega_{\text{tot}}} = \frac{2^N \binom{N}{M}}{\binom{2N}{N+M}} = \frac{2^N \frac{N!}{M!(N-M)!}}{\frac{(2N)!}{(N+M)!(N-M)!}} \\ = 2^N \frac{N!(N+M)!}{(2N)!M!} \end{aligned}$$