Solutions to exercises week 36
FYS2160

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Schroeder 2.7

Representing $N = 4$ oscillators and $q = 2$ energy units in terms of lines and dots, we need: 3 lines (representing partitions between the oscillators) and 2 dots (one dot representing one energy unit). There are 10 possible arrangements of 3 lines and 2 dots:

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Schroeder 2.8

Consider two weakly coupled Einstein solids, A and B, with number of oscillators $N_A = N_B = 10 \Rightarrow N = N_A + N_B = 20$ and energy units $q = q_A + q_B = 20 \Rightarrow q_B = 20 - q_A$.

a)

There are in total 21 macrostates available to the system, which can be labeled by $q_A = 0, 1, 2, ..., 20$, where $q_B = 20 - q_A$.

b)

Total number of available microstates in the system:

$$\Omega(N = 20, q = 20) = \binom{20 + 20 - 1}{20} = \frac{39!}{20!19!} \approx 6.89 \times 10^{10}.$$

c)

The probability of finding all the energy in system A, when the two subsystems are in thermal equilibrium, is

$$P(q_A = 20) = \frac{\Omega(q_A = 20)}{\Omega_{\text{tot}}} = \frac{\Omega_A(N_A = 10, q_A = 20) \cdot \Omega_B(N_B = 10, q_B = 0)}{\Omega(N = 20, q = 20)}.$$

The multiplicities of the subsystems are

$$\Omega_A(N_A = 10, q_A = 20) = \binom{20 + 10 - 1}{20} = \frac{29!}{20!19!} \approx 1.00 \times 10^7,$$

$$\Omega_B(N_B = 10, q_B = 0) = \binom{0 + 10 - 1}{0} = \frac{9!}{0!9!} = 1,$$

which gives a probability of

$$P(q_A = 20) = \frac{\Omega(q_A = 20)}{\Omega_{\text{tot}}} = \frac{1.00 \times 10^7}{6.89 \times 10^{10}} \approx 1.45 \times 10^{-4}.$$

d)

The probability of finding exactly half the energy in solid A, is

$$\begin{aligned} P(q_A = 10) &= \frac{\Omega(q_A = 10)}{\Omega_{\text{tot}}} \\ &= \frac{\Omega_A(N_A = 10, q_A = 10) \cdot \Omega_B(N_B = 10, q_B = 10)}{\Omega(N = 20, q = 20)}. \end{aligned}$$

The multiplicities of the subsystems are in this case the same:

$$\Omega_A(N_A = 10, q_A = 10) = \Omega_B(N_B = 10, q_B = 10) = \binom{10 + 10 - 1}{10} = \frac{19!}{10!9!}$$
$$= 92378,$$

which gives a probability of

$$P(q_A = 10) = \frac{\Omega(q_A = 10)}{\Omega_{\text{tot}}} = \frac{92378^2}{6.89 \times 10^{10}} \approx 0.124.$$

e)

We have found that it is about a thousand times more likely to find the system with the energy distributed equally between the two solids than finding all the energy in solid A. Hence, if we start out with all energy in solid A, we would after some time expect to find the system in a state with the energy more equally distributed. And conversely, if the system starts out with equal amounts of energy in the two solids, we would at a later point not expect to find all energy in solid A. So if the energy to a large degree is unevenly distributed when the solids are put in thermal contact, we expect the energy to "irreversibly" distribute itself more evenly between the two solids.

Schroeder 2.22

a)

There are $2N + 1$ macrostates in total.

b)

Using the result from Problem 2.18 in Schroeder, and treating the combined system as one Einstein solid with $2N$ oscillators and $2N$ energy units, we can estimate the total multiplicity of the combined system as approximately

$$\Omega(2N, 2N) \approx \frac{\left(\frac{2N+2N}{2N}\right)^{2N}\left(\frac{2N+2N}{2N}\right)^{2N}}{\sqrt{2\pi 2N(2N+2N)/2N}} = \frac{2^{2N}2^{2N}}{\sqrt{8\pi N}} = \frac{2^{4N}}{\sqrt{8\pi N}}.$$

c)

The most likely macrostate is the one where the energy is equally distributed between the two solids, that is, both solids have N energy units, and the combined multiplicity of this state is then

$$\begin{aligned}\Omega(2N, N) = \Omega_A(N, N) \cdot \Omega_B(N, N) \quad &\approx \left[\frac{\left(\frac{N+N}{N}\right)^{N}\left(\frac{N+N}{N}\right)^{N}}{\sqrt{2\pi N(N+N)/N}}\right]^2 = \left[\frac{2^N 2^N}{\sqrt{4\pi N}}\right]^2 \\ &= \frac{2^{4N}}{4\pi N}.\end{aligned}$$

d)

We crudely estimate the peak of the multilicity, as a function of q, to be a rectangle, with the height h given by the result of c) and the area A by the result in b). The width w of this peak is then

$$w = \frac{A}{h} = \frac{2^{4N}/\sqrt{8\pi N}}{2^{4N}/4\pi N} = \frac{4\pi N}{\sqrt{8\pi N}} = \sqrt{2\pi N}.$$

The width of this peak relative to the width of the total range of q-values is

$$\frac{w}{2N+1} = \frac{\sqrt{2\pi N}}{2N+1} \xrightarrow[N\to\infty]{} \sqrt{\frac{\pi}{2N}} \sim \frac{1}{\sqrt{N}}.$$

So the fraction of the macrostates with reasonably large probabilities goes approximately as $1/\sqrt{N}$. For $N = 10^{23}$ we have

$$\frac{1}{\sqrt{10^{23}}} \approx 3 \times 10^{-12},$$

which is less than one part in 100 billion (10^{-11}).

Schroeder 2.27

The multiplicity of a monatomic ideal gas can be expressed as

$$\Omega(U, V, N) = f(N)V^N U^{3N/2},$$

where $f(N)$ is some function of N. The probability of all molecules being in the leftmost 99 % of the container is then

$$P(U, 0.99V, N) = \frac{\Omega(U, 0.99V, N)}{\Omega(U, V, N)} = \frac{(0.99V)^N}{V^N} = 0.99^N.$$

For a container with 100, 10 000 and 10^{23} molecules, respectively, the probability of finding all molecules in the leftmost 99 % is

$$P(U, 0.99V, N = 100) = 0.99^{100} \approx 0.37,$$
$$P(U, 0.99V, N = 10^4) = 0.99^{10^4} \approx 2.2 \times 10^{-44},$$
$$P(U, 0.99V, N = 10^{23}) = 0.99^{10^{23}} \approx 10^{-10^{21}}.$$