Solutions to exercises week 37

FYS2160

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November 7, 2017

Schroeder 2.18

$\Omega(N,q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!}$

Use that $(N-1!) = N!/N$:

$\Omega(N,q) = \frac{N}{q+N} \cdot \frac{(q+N)!}{q!N!}$

Use Stirling's approximation ($N! \approx N^N e^{-N}\sqrt{2\pi N}$):

$\Omega(N,q) \approx \frac{N}{q+N} \cdot \frac{(q+N)^{q+N}e^{-(q+N)}\sqrt{2\pi(q+N)}}{q^q e^{-q}\sqrt{2\pi q}\ N^N e^{-N}\sqrt{2\pi N}} = \frac{(q+N)^{(q+N)}}{q^q N^N}\sqrt{\frac{N}{2\pi q(q+N)}}$

Write $(q+N)^{(q+N)}$ as $(q+N)^q(q+N)^N$:

$\Omega(N,q) \approx \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N \sqrt{\frac{N}{2\pi q(q+N)}}$

Schroeder 2.24

a)

Most likely macrostate is $N_\uparrow = N_\downarrow = N/2$:

$\Omega_{\max} = \frac{N!}{N_\uparrow! N_\downarrow!} = \frac{N!}{(\frac{N}{2}!)^2} \approx \frac{N^N e^{-N}\sqrt{2\pi N}}{((\frac{N}{2})^{N/2}e^{-N/2}\sqrt{2\pi N/2})^2} = 2^N\sqrt{\frac{2}{\pi N}}$

b)

By Stirling's approximation we have:

$\Omega \approx \frac{N^N}{N_\uparrow^{N_\uparrow} N_\downarrow^{N_\downarrow}}\sqrt{\frac{N}{2\pi N_\uparrow N_\downarrow}}$

Using $N_\uparrow = (N/2) + x$ and $N_\downarrow = (N/2) - x$ we get:

$$\Omega \approx \frac{N^N}{(\frac{N}{2}+x)^{\frac{N}{2}+x}(\frac{N}{2}-x)^{\frac{N}{2}-x}}\sqrt{\frac{N}{2\pi(\frac{N}{2}+x)(\frac{N}{2}-x)}} = \frac{N^N}{[(\frac{N}{2})^2-x^2]^{N/2}(\frac{N}{2}+x)^{x}(\frac{N}{2}-x)^{-x}}\sqrt{\frac{N}{2\pi\ [(\frac{N}{2})^2-x^2]}}$$

Take the logarithm:

$$\ln\Omega = N\ln N - \frac{N}{2}\ln\left[(\frac{N}{2})^2 - x^2\right] - x\ln(\frac{N}{2}+x) + x\ln(\frac{N}{2}-x)$$
$$+\ln\sqrt{\frac{N}{2\pi}} - \frac{1}{2}\ln\left[(\frac{N}{2})^2 - x^2\right]$$

Assume $x << N$ and expand logarithms:

$$\ln\left[(\frac{N}{2})^2 - x^2\right] = \ln(\frac{N}{2})^2 + \ln\left[1 - (\frac{2x}{N})^2\right] \approx 2\ln(\frac{N}{2}) - (\frac{2x}{N})^2$$

and

$$\ln\left(\frac{N}{2} \pm x\right) = \ln(\frac{N}{2}) + \ln\left[1 \pm \frac{2x}{N}\right] \approx \ln(\frac{N}{2}) \pm \frac{2x}{N}$$

Applying to $\ln\Omega$:

$$\ln\Omega = N\ln N - N\ln\frac{N}{2} + \frac{2x^2}{N} - x\ln\frac{N}{2} - \frac{2x^2}{N} + x\ln\frac{N}{2} - \frac{2x^2}{N} + \ln\sqrt{\frac{N}{2\pi}} - \ln\frac{N}{2} + \frac{2x^2}{N^2}$$
$$= N\ln 2 - \frac{2x^2}{N} - \ln\sqrt{\frac{2}{\pi N}} - \frac{2x^2}{N^2}$$

We can neglect the last term and exponentiate to get:

$$\Omega = 2^N\sqrt{\frac{2}{\pi N}}e^{-2x^2/N} \qquad \text{(for } x << N\text{)}.$$

This is a Gaussian function with a peak at $x = 0$ with the value from **a)**.

c)

The function Ω falls of to $1/e$ of the peak for $x = \sqrt{N/2}$. Width of peak is twice of this, giving us a width of $\sqrt{2N}$.

d)

$N = 10^6$. Half width is $\sqrt{5 \cdot 10^5} \approx 700$.

An excess of 1000 heads or tails is a little beyond the half width, and such a result would not be surprising.

An excess of 10 000 heads is far beyond the half width and the peak. At this point the multiplicity falls of by $e^{-200} \approx 10^{-87}$, and this would be surprising and an indication that the coins are not fair.

Schroeder 2.25

a)

Most likely to end where you start. Equally likely to go back and forward for each step randomly.

b)

Can use the same distribution we got in 2.24, a Gaussian of the form $e^{-2x^2/N}$, where N is number of steps and x is the excess of forward steps over $N/2$. When we introduce step length l we get the distribution on the form $e^{-x^2/2l^2N}$, where $x \to x/2l$ is the distance travelled from where we started.

We get the half width (multiplicity falls of to $1/e$) at $\sqrt{2N}l$, or $\sqrt{2N}$ step lengths. For $N =$10 000 we get a half width of 140 steps, so there is a good chance to end up withing 140 steps in either direction from where we started. Getting farther away then 500 steps is negligible.

c)

Mean free path: $l \approx 150$ nm

Average collision time: $\bar{\Delta t} \approx 3 \times 10^{-10}$ s

In 1 second we approximately $N = 3 \times 10^9$ steps (collisions).

Expected net distance travelled: $\sqrt{2N} = 80\ 000$ steps, which corresponds to 12×10^6 nm, or 12 mm.

For longer times the number of steps N increase in proportion to the time t, so average distance travelled increases in proportion to $\sqrt{t}$.

For higher temperatures:

- Mean free path which is proportional to V/N would increase proportional to T ($PV = nRT$).
- Molecules move faster in proportion to $\sqrt{T}$ ($v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$).
- Collision time $\bar{\Delta t} = l/\bar{v}$ increase in proportion to $\sqrt{T}$.

- Number of steps increase in proportion to $1/\sqrt{T}$.
- Expected net steps travelled increase proportional to $\sqrt{N} \propto T^{-1/4}$.
- Expected net distance, which is expected net steps multiplied with mean free path, increase in proportion to $T^{3/4}$

Schroeder 3.1

For $q_A = 1$:

$T_A = (\frac{\partial U_A}{\partial S_A})_{N,V} = \frac{2\epsilon - 0\epsilon}{10.7k - 0k} = 0.19\frac{\epsilon}{k} = 220K$

$T_B = (\frac{\partial U_B}{\partial S_B})_{N,V} = \frac{100\epsilon - 98\epsilon}{187.5k - 185.3k} = 0.91\frac{\epsilon}{k} = 1060K$

For $q_A = 60$:

$T_A = (\frac{\partial U_A}{\partial S_A})_{N,V} = \frac{61\epsilon - 59\epsilon}{160.9k - 157.4k} = 0.57\frac{\epsilon}{k} = 660K$

$T_B = (\frac{\partial U_B}{\partial S_B})_{N,V} = \frac{41\epsilon - 39\epsilon}{107.0k - 103.5k} = 0.57\frac{\epsilon}{k} = 660K$

Schroeder 3.3

Initially: $\frac{\delta S_A}{\delta U_A} > \frac{\delta S_B}{\delta U_B}$

U_A will increase, U_B will decrease until:

$\frac{\delta S_A}{\delta U_A} = \frac{\delta S_B}{\delta U_B}$

Schroeder 3.4

A "miserly" system A can be in thermal equilibrium with another system B. If B is "miserly" a small flow of energy from B to A give increase in temperature of B and decrease in temperature of A causing a run-away effect, with more and more energy flowing from B to A.

If B is a large "reservoir" where the temperature doesn't change significantly. With a small transfer of energy from B to A leads to decrease in temperature of A causing a run-away effect again.

If B is a "normal" system that is sufficiently small (low heat capacity) such that a spontaneous transfer of energy from B to A causes B to cool off more than A does. In this case A will become a bit hotter than B and the energy will spontaneously flow back giving us a stable thermal equilibrium.