Weekly exercise W44: FYS2160, Thermodynamics and statistical physics, Fall 2022

1 Problems from Schroeder

Schroeder 7.11 Solution:
The probability of a state being occupied is given by the Fermi-Dirac dis- tribution function: $1/(e^{(\epsilon-\mu)/kT}+1)$ *. At room temperature,* kT *= .026 eV, so the probabilities are:*

(a)
$\epsilon-\mu$ *= -1 eV in this case, giving* $P = 1/(e^{-1/0.026}+1) = (1+2x10^{-17})^{-1} = 1\text{-}2x10^{-17} \approx 1$
(b)
$\epsilon-\mu$ *= -0.01 eV in this case, giving* $P = 1/(e^{-0.01/0.026}+1) = 1/1.68 = 0.59$
(c)
$\epsilon-\mu$ *= 0 eV in this case, giving* $P = 1/(e^{0}+1) = 1/2 = 0.5$
(d)
$\epsilon-\mu$ *= 0.01 eV in this case, giving* $P = 1/(e^{0.01/0.026}+1) = 1/2.47 = 0.41$
(e)
$\epsilon-\mu$ *= 1 eV in this case, giving* $P = 1/(e^{1/0.026}+1) = (5x10^{16})^{-1} = 2x10^{-17}$

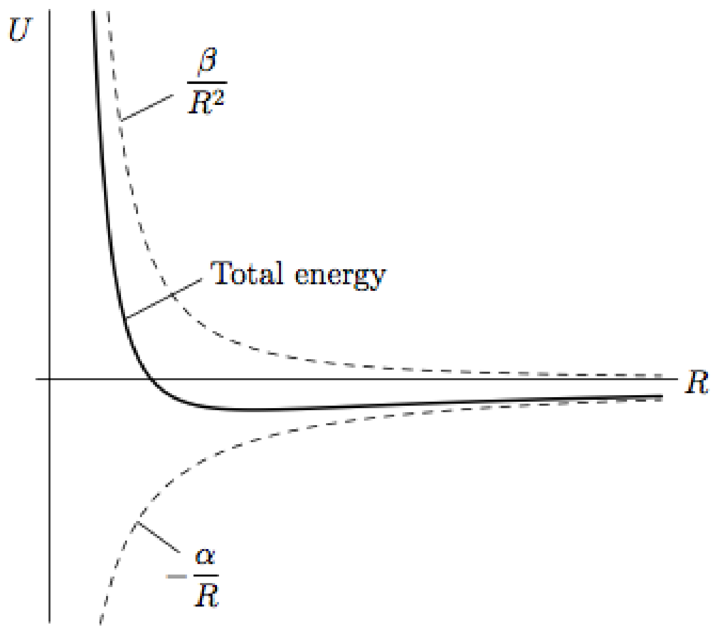
Schroeder 7.15 Solution:

For a system of particles obeying the Boltzmann distribution, the total number of particles should be:
$N= \sum_s \bar{n}_{Boltzmann} = \sum_s e^{-(\epsilon_s-\mu)/kT} = e^{\mu/kT}\sum_s e^{-\epsilon_s/kT}$
But the sum in the last expression is just the single-particle partition function, therefore,
$\frac{N}{Z_1} = e^{\mu/kT}$,
$\mu = kT ln\frac{N}{Z_1} = -kT ln\frac{Z_1}{N}$.

Schroeder 7.23 Solution:

(a)
We want to get dimensions of U(Nm), using G(Nm²/kg²):
$Nm = (Nm^2/kg^2)\cdot (kg^2/m)$
corresponding to:
$U = -GM^2/R$
We put in a minus sign since gravity is attractive: We would have to add energy to disassemble the sphere, moving the parts infinitely far apart where they have zero potential energy.
(b)
The total energy of the degenerate electron gas is:
$U_{kinetic} = \frac{3}{5}N\epsilon_F = \frac{3}{5}N\cdot\frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3}$
where N is the number of electrons. If the star contains one proton (mass m_p*) and one neutron (mass* $\approx m_p$*) for each electron, then* $N = M/2m_p$*. Plugging in* $4\pi R^3/3$ *for the volume then gives:*
$U_{kinetic} = \frac{3h^2}{40m_e}\left(\frac{M}{2m_p}\right)^{5/3}\left(\frac{9}{4\pi^2R^3}\right)^{2/3} = 0.0088\frac{h^2M^{5/3}}{m_e m_p^{5/3}R^2}$
(c)

The gravitational energy of the star is proportional to -1/R, while the kinetic energy of the electrons is proportional to 1/R2 from the equations found above. Here's a sketch of these functions and their sum:



To find the minimum in the total energy, set the derivative equal to zero:

$0 = \frac{d}{dR}(-\alpha/R + \beta/R^2) = \alpha/R^2 - \beta/R^3 = \frac{1}{R^2}(\alpha - 2\beta/R$

The equilibrium radius is therefore at

$R = 2\beta/\alpha = 0.029\frac{h^2}{Gm_e m_p^{5/3}}\frac{1}{M^{1/3}}$

A white dwarf star with a larger mass has a smaller equilibrium radius. This does make sense, because adding mass creates more gravitational attraction, allowing the gravitational energy to decrease more than the kinetic energy increases as the star contracts

(d)

Using the above equation for one solar mass gives R = 7.2 x 10^6 m = 7200 km, which is just sligthly larger than the earth. The density is:

$\rho = \frac{M}{4\pi R^3/3} = \frac{2\cdot 10^{30}kg}{(4/3)\pi(7.2\cdot 10^6 m)^3} = 1.3\cdot 10^9 kg/m^3$

Which is 1.3 million times the density of water.

(e)

The fermi energy is:

$\epsilon_F = \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3}\frac{1}{R^2} = 3.1\cdot 10^{-14}J = 1.9\cdot 10^5 eV$

Giving the Fermi temperature:

$T_F = \epsilon_F/k = 2.3\cdot 10^9 K$

This is more than a hundred times hotter than the center of the sun. It seems unlikely that the actual temperature of a white dwarf star would be anywhere near this high. In other words, the thermal energy of the electrons is almost certainly much smaller than the kinetic energy they have even at T = 0. For the purposes of the energy calculations in this problem, therefore, simply neglecting the thermal energy and setting T = 0 is probably an excellent approximation.

(f)

If the electrons are ultra-relativistic, we can use the formulas derived in the previous problem for the Fermi energy and the total kinetic energy:

$U_{kinetic} = \frac{3}{4} N \epsilon_F = \frac{3}{4} N \cdot \frac{hc}{2} \left(\frac{3N}{\pi V}\right)^{1/3} = 0.091 hc \left(\frac{M}{m_p}\right)^{4/3} \frac{1}{R}$

The important feature of this formula is that it is proportional to 1/R, not 1/R^2. When we add the gravitational potential energy, which is proportional to -1/R, we get a total energy function with no stable minimum. Instead, depending on which coefficient is larger, the total energy is simply proportional to either +1/R or -1/R. Therefore the "star" will either expand to infinite radius or collapse to zero radius.

(g)

First note that the coe cient of the gravitational energy is proportional to M2, while that of the kinetic energy is proportional to only $M^{4/3}$, so the star will collapse rather than expand if its mass is sufficiently large. The crossover from expansion to collapse occurs when the coefficients are equal, that is, when

$0.091 hc \left(\frac{M}{m_p}\right)^{4/3} = \frac{3}{5} G M^2$

giving $M = 3.4 \cdot 10^{30} kg$

that is, a little under twice the sun's mass. However, the star won't be relativistic to begin with unless the average kinetic energy of the electrons is comparable to their rest energy, $mc^2 = 5 \cdot 10^5$ *eV. For the sun's mass, the average electron energy (0.6ϵ_F) is only 1.2·10^5 eV, too low by a factor of about 4.4. This indicates that a one-solar- mass white dwarf is probably stable, but it's still close enough to being relativistic that we shouldn't expect the nonrelativistic approximation to be terribly accurate. Meanwhile, looking back at part (e), we see that the Fermi energy is proportional to* $(M/R^3)^{2/3} \propto (M^2)^{2/3} = M^{4/3}$*. Therefore, to increase the Fermi energy by a factor of 4.4, we'd have to increase the mass by only a factor of about 3. Conclusion: A white dwarf star with a mass greater than about three times the sun's mass will be relativistic and hence unstable, collapsing to zero radius (unless it first converts into some other form of matter).*