Exercise week 37: FYS2160, Thermodynamics and statistical physics

Problems from Schroeder: 2.28, 2.30, 2.32, 2.38, 2.42, 3.5, 3.7, 3.13

Solution:
Problem 2.28:
There are 52 cards that could be on top, and for each of these choices there are 51 possibilities for the next card, then 50 for the next and so forth down to the 1 choice for the bottom card. The total number of card configurations is the permutation of 52 cards: $\Omega = 52!$. If all states are accessible, then the configurational entropy is

$$S = k \ln \Omega = k \ln 52! = 2.16 \times 10^{-21} J/K$$

in SI units and

$$\frac{S}{k} = \ln \Omega = \ln 52! = 156$$

Solution:
Problem 2.30:
(a) From problem 2.22b we know that the total number of microstates is

$$\Omega_{\text{total}} = \frac{2^{4N}}{\sqrt{8\pi N}}$$

thus giving the entropy

$$\frac{S}{k} = \ln \frac{2^{4N}}{\sqrt{8\pi N}} = 4N \ln 2 - \frac{1}{2} \ln 8\pi N = 2.77 \times 10^{23} - 28.1$$

for $N = 10^{23}$.
(b) From problem 2.22c we know that the multiplicity of the most probable macrostate is

$$\Omega_{\text{total}} = \frac{2^{4N}}{4\pi N}$$

thus giving the entropy

$$\frac{S}{k} = \ln \frac{2^{4N}}{4\pi N} = 4N \ln 2 - \ln 4\pi N 2.77 \times 10^{23} - 55.5$$

for $N = 10^{23}$.
(c) The difference in the two entropies is only $55.5 - 28.1 = 27.4$ in units of the Boltzmann's constant, k. This is negligible compared to the main contribution $4N \ln 2$.

(b) Inserting the partition causes the entropy to decrease by about 27 units out of $4N \ln 2$ units, which again is a negligible effect.

Solution:
Problem 2.32:
Taking the logarithm of the result of problem 2.26 and using Stirling's approximation:

$$\frac{S}{k} = \ln\left(\frac{1}{(N!)^2}\left(\frac{2\pi mUA}{h^2}\right)^N\right) = N\ln\left(\frac{2\pi mUA}{h^2}\right) - 2(N\ln N - N)$$

Solution:
Problem 2.38:
Multiplicity due to mixing:

$$\Omega_{mixing} = \frac{N!}{N_A!N_B!}$$

hence the change in entropy of mixing is $\Delta S_{mixing} = k\ln\Omega_{mixing}$. Assuming that both N_A and N_B are large so we can use Stirling's approximation

$$S_{mixing} \approx k[N\ln N - N_A\ln N_A - N_B\ln N_B]$$

Expressing N_A and N_B in terms of the concentration $x = N_B/N$ of component B: $N_A = (1-x)N$ and $N_B = xN$,

$$S_{mixing} = -Nk[x\ln x + (1-x)\ln(1-x)]$$

Solution:
Problem 2.42:
(a) In the SI system, the units of G are $N\cdot m^2/kg^2$. $1N = 1kg\cdot m/s^2$, so the units of G can also be written as $m^3/kg\cdot s^2$. Mass M has units of kg and the speed of light c has units of m/s. We can get meters by combining G or c to cancel seconds, as G/c^2, which has units of m^2/kg. Hence the typical radius is $\sim MG/c^2$.
(b) The entropy of a system is of the same order as the number of particles in the system, N. If we compress it to form a black hole, the 2nd law requires that the entropy of the black hole is still at least of order N. But since the end state is the same whether we start with a lot of particles or a few, the final entropy must in fact be of the same order as the maximum N, the largest possible number of particles that it could have been formed from.
(c) Suppose we have N photons, each with wavelength equal to the size of the black hole: $\lambda = GM/c^2$. The photon's energy is $\epsilon = hc/\lambda$, and the total energy of the photon gas equals Mc^2

$$Mc^2 = N\epsilon = \frac{Nhc^3}{GM} \to N = \frac{GM^2}{hc}$$

and the entropy is of the same order as N

$$S \sim k \frac{GM^2}{hc}$$

(d) For a one solar mass black hole

$$\frac{S}{k} = \frac{8\pi^2(6.67 \times 10^{-11} N \cdot m^2/kg^2)^2(2 \times 10^{30} kg)^2}{(6.63 \times 10^{-34} J \cdot s)(3 \times 10^8 m/s)} = 1.06 \times 10^{77}$$

For comparison, an ordinary star has a number of particles of the order of 10^{57} particles. Thus implying an entropy of order $10^{57}k$, giving the solar mass black hole an entropy of 20 orders of magnitude greater than a comparable sun. In other words, black holes has an enormous entropy compared to comparable ordinary stellar objects.

Solution:
Problem 3.5:
From problem 2.17 we know that the multiplicity of an Einstein solid in the low T limit, $q \ll N$ is

$$\Omega(q, N) = \left(\frac{eN}{q}\right)^q$$

thus giving the corresponding entropy

$$S(q, N) = k \ln \left(\frac{eN}{q}\right)^q = kq[\ln N - \ln q + 1]$$

The internal energy is $U = q\epsilon$, where ϵ is the size of the each energy unit. This allows us to express the entropy as function of internal energy:

$$S(U, N) = \frac{kU}{\epsilon}[\ln N - \ln U + \ln \epsilon + 1]$$

The inverse of temperature is then

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N = \frac{kU}{\epsilon} \ln \frac{N\epsilon}{U}$$

By reordering the expression and exponentiating both sides, we find the internal energy of the Einstein solid as a function of temperature for a given N:

$$U = N\epsilon e^{-\epsilon/(kT)}$$

Solution:
Problem 3.7:
From problem 2.24 we know that the entropy of a black hole is

$$S(U) = \frac{8\pi^2 GM^2 k}{hc} = \frac{8\pi^2 GU^2 k}{hc^5}$$

where $U = Mc^2$ is its internal energy. Hence, the temperature of the black hole is

$$T = \left(\frac{\partial S}{\partial U}\right)^{-1} = \frac{hc^3}{16\pi^2 GMk}$$

for $M = 2 \times 10^{30} kg$, the temperature is evaluated to

$$T = 6.1 \times 10^{-8} K$$

giving the back hole a very low observable temperature. Figure 1 shows a sketch of the entropy of the system.

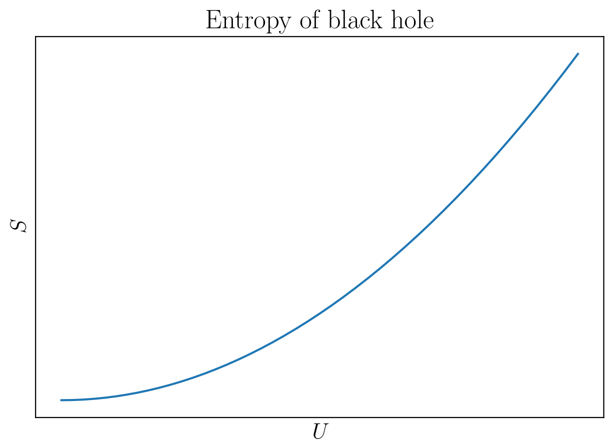

Figure 1: A sketch of the entropy of a black hole as a function of temperature.

Solution:
Problem 3.13:
(a) Assuming an average of 8 hours of sunlight per day, then 1 m^2 of Earth's surface receives in 1 year a total energy of

$$(1000 J/s)(3600 s/hr)(8 \text{ hrs/day })(365 \text{ days}) = 1.05 \times 10^{10} J$$

The entropy gained by the Earth upon receiving this heat:

$$\Delta S_{Earth} = \frac{Q}{T} = 3.5 \times 10^7 J/K$$

The entropy lost by the Sun is 20 times smaller than this, since the sun's surface is 20 times hotter than the Earth's,

$$\Delta S_{Sun} = -0.175 \times 10^7 J/K$$

The total change in entropy (rounded off):

$$\Delta S_{Sun} + \Delta S_{Earth} \approx 3 \times 10^7 J/K$$

(b) The net reduction in entropy is

$$Nk = nR \approx (1000 \text{ moles })(8.3 J/mol \cdot K) \sim 10^4 J/K,$$

about 3000 times less than the entropy created by sunlight warming the ground. The growth of the grass merely reduces the increase in entropy by a tiny fraction of a percent and is thus negligible.