

## 1 Problems from Schroeder

### Schroeder 7.37 Solution:

**Schroeder 7.37** To find the maximum of the Planck function  $x^3/(e^x - 1)$ , you could just plug in numbers until it becomes clear that  $x = 2.82$  gives a larger value than any other  $x$ . Or you could use an algorithm to solve for the minimum of the function.

You could also derive the function and set it to 0, leaving you with the equation  $e^{-x} = 1 - x/3$ , which you again would need to find the root of numerically.

### Schroeder 7.39 Solution:

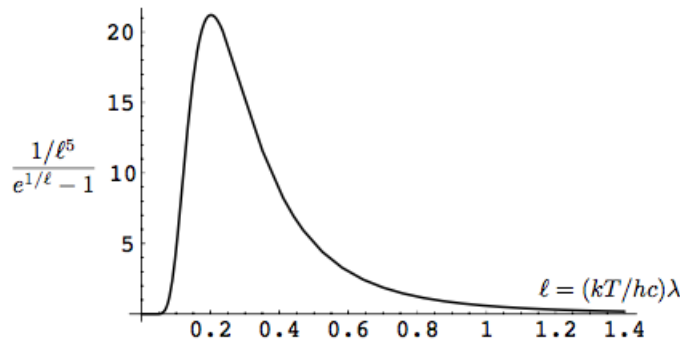
**Schroeder 7.39** The relation  $\lambda = hc/\epsilon$  implies  $\epsilon = hc/\lambda$  and  $d\epsilon = -(hc/\lambda^2)d\lambda$ . Therefore, equation 7.83 becomes:

$$\frac{U}{V} = \frac{8\pi}{(hc)^3} \int_0^\infty \frac{\epsilon^3}{e^{\epsilon/kT} - 1} d\epsilon = -\frac{8\pi}{(hc)^3} (hc)^4 \int_\infty^0 \frac{1/\lambda^5}{e^{hc/kT\lambda} - 1} d\lambda = 8\pi (hc) \int_0^\infty \frac{1/\lambda^5}{e^{hc/kT\lambda} - 1} d\lambda.$$

The function whose integral gives the energy density is from that:

$$u(\lambda) = \frac{8\pi hc/\lambda^5}{e^{hc/kT\lambda} - 1}$$

below is the plot of the function with dimensionless variables:



Although this curve may look superficially similar to Figure 7.19, it differs significantly: It goes to zero exponentially as  $l$  goes to 0, and falls off in proportion to  $1/L^4$  as  $l$  goes to infinity. Notice also that the peak of the curve is at approximately  $l = 0.2$ . Therefore, as a function of wavelength, the Planck spectrum peaks at the value  $l = (0.2)(hc/kT) = hc/(5kT)$ . This is not the same point where the spectrum peaks as a function of photon energy (or frequency); it is lower in wavelength (or higher in energy) by a factor of  $5/2.82 = 1.76$ . How can this be? The difference occurs because both this spectrum and the one plotted in Figure 7.19 are functions whose purpose in life is to be integrated. Either formula will give the same answer when you integrate it appropriately between any two points. However, one gives the energy density per unit photon energy, while the other gives

the energy density per unit wavelength, and the units of photon energy and wavelength depend on each other in a nonlinear way. For instance, a one-unit photon energy range, say a range of 1 eV, corresponds to a larger range of wavelengths if it's from 2 eV to 3 eV than if it's from 200 eV to 201 eV. Therefore, when we convert the photon-energy spectrum to the wavelength spectrum, we shift the curve upward at short wavelengths (where a unit of wavelength includes a relatively large range of photon energies) and downward at long wavelengths (where a unit of wavelength includes a relatively small range of photon energies).

### Schroeder 7.58 Solution:

**Problem 7.58.** The density of copper (which I looked up in an introductory physics textbook) is  $8.93 \text{ g/cm}^3$ , and the atomic mass (which I got off a periodic table) is  $63.5 \text{ g/mol}$ . Let's consider a chunk of copper containing one mole of atoms. Then the mass is  $63.5 \text{ g}$ , and the volume is

$$V = \frac{\text{mass}}{\text{density}} = \frac{63.5 \text{ g}}{8.93 \text{ g/cm}^3} = 7.11 \times 10^{-6} \text{ m}^3.$$

The theoretical Debye temperature is therefore

$$T_D = \frac{hc_s}{2k} \left( \frac{6N}{\pi V} \right)^{1/3} = \frac{(6.63 \times 10^{-34} \text{ J}\cdot\text{s})(3560 \text{ m/s})}{2(1.38 \times 10^{-23} \text{ J/K})} \left( \frac{6(6.02 \times 10^{23})}{\pi(7.11 \times 10^{-6} \text{ m}^3)} \right)^{1/3} = 466 \text{ K}.$$

Meanwhile, the slope of the line for copper in Figure 7.28 is roughly

$$\text{slope} = \frac{0.9 \text{ mJ/K}^2}{18 \text{ K}^2} = 5 \times 10^{-5} \text{ J/K}^4.$$

According to equation 7.116, this slope should be equal to  $12\pi^4 Nk/5T_D^3$ . Solving for  $T_D$  then gives us

$$T_D = \left( \frac{12\pi^4 Nk}{5(\text{slope})} \right)^{1/3} = \left( \frac{12\pi^4 (8.315 \text{ J/K})}{5(5 \times 10^{-5} \text{ J/K}^4)} \right)^{1/3} = 340 \text{ K}.$$

So the theoretical prediction is too large by nearly 40%. I suspect that the value used here for the speed of sound applies only to longitudinal waves, with transverse waves traveling considerably slower and therefore lowering the effective average Debye temperature.

### Schroeder 7.60 Solution:

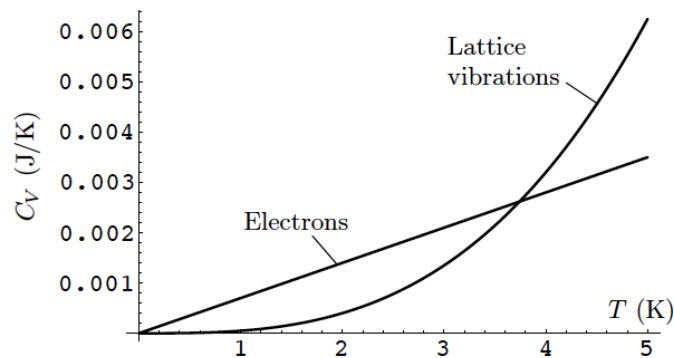
**Problem 7.60.** At low temperatures, the electronic contribution to the heat capacity is linear in temperature while the contribution of lattice vibrations is cubic. To draw them correctly on the same graph, it's easiest to first solve for the temperature at which they are equal. From equation 7.116, this is where

$$\gamma T = \alpha T^3, \quad \text{or} \quad T = \sqrt{\frac{\gamma}{\alpha}},$$

where  $\alpha$  is an abbreviation for  $12\pi^4 Nk/5T_D^3$ , the slope of the graph plotted in Figure 7.28. From the data for copper plotted in the figure, this slope appears to be roughly  $(0.9 \text{ mJ/K}^2)/(18 \text{ K}^2) = 5 \times 10^{-5} \text{ J/K}^4$ , while  $\gamma$ , the intercept, appears to be roughly  $0.7 \text{ mJ/K}^2$ . Therefore the temperature at which the two contributions are equal should be

$$T = \sqrt{\frac{7 \times 10^{-4} \text{ J/K}^2}{5 \times 10^{-5} \text{ J/K}^4}} = \sqrt{14 \text{ K}^2} = 3.7 \text{ K}.$$

At this temperature, each of the contributions to the heat capacity is equal to  $\gamma T = (7 \times 10^{-4} \text{ J/K}^2)(3.7 \text{ K}) = 0.0026 \text{ J/K}$ . Here, then, is a plot of the two contributions separately:



**Schroeder 7.61 Solution:**

**Problem 7.61.** If we repeat the derivation on pages 308 through 311 for the case of a liquid, the only thing that changes is the number of polarization states for each triplet  $(n_x, n_y, n_z)$ : now there is only one polarization rather than three. This change has no effect on the volume of  $n$ -space that is summed over, and therefore has no effect on the formula  $n_{\max} = (6N/\pi)^{1/3}$  or on equation 7.111 for the Debye temperature:

$$T_D = \frac{hc_s}{2k} \left( \frac{6N}{\pi V} \right)^{1/3}.$$

To evaluate this expression we need to know the ratio  $N/V$ . Let's take  $N$  to be Avogadro's number, so that the mass of the sample is 4 g. At a density of 0.145 g/cm<sup>3</sup>, this value implies a volume of 27.6 cm<sup>3</sup> or  $2.76 \times 10^{-5}$  m<sup>3</sup>. The predicted Debye temperature is therefore

$$T_D = \frac{(6.63 \times 10^{-32} \text{ J}\cdot\text{s})(238 \text{ m/s})}{2(1.38 \times 10^{-23} \text{ J/K})} \left( \frac{6(6.02 \times 10^{23})}{\pi(2.76 \times 10^{-5} \text{ m}^3)} \right)^{1/3} = 19.8 \text{ K}.$$

What does change in the derivation in the text is the numerical factor multiplying the energy (and the heat capacity): The factor of 3 in equation 7.106 disappears, so each expression for  $U$  or  $C_V$  from there on should be divided by 3 for the case of a liquid. The heat capacity in the low-temperature limit is therefore 1/3 times the formula in equation 7.115:

$$\frac{C_V}{Nk} = \frac{4\pi^4}{5} \left( \frac{T}{T_D} \right)^3.$$

The cube root of  $5/4\pi^4$  is 0.234, so this is the same as

$$\frac{C_V}{Nk} = \left( \frac{T}{(0.234)T_D} \right)^3 = \left( \frac{T}{4.64 \text{ K}} \right)^3,$$

in almost perfect agreement with the measured behavior.