Symmetry and degeneracy[1]

Symmetry. Let us first define a symmetry. Consider a state at $t = 0$, $|\psi(0)\rangle$, and apply a unitary transformation T to this state. The result is a new state $|\psi'(0)\rangle$,

$$|\psi'(0)\rangle = T|\psi(0)\rangle. \tag{1}$$

We will now consider the time-development of both states $|\psi(0)\rangle$ and $|\psi'(0)\rangle$. The time-development of each state is given by applying the operator $U = e^{-iHt/\hbar}$ which gives

$$|\psi'(t)\rangle = e^{-iHt/\hbar}|\psi'(0)\rangle \quad \text{and} \quad |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle. \tag{2}$$

The transformation T is a *symmetry* (transformation) if

$$|\psi'(t)\rangle = T|\psi(t)\rangle \tag{3}$$

for all $|\psi(0)\rangle$ and times t. Thus it makes no difference whether we study the time-evolution of the symmetry-transformed state or the original state, they will always (at any time t) be related by a symmetry transformation. Another way of stating this is that we will get at the same state if we 'transform first and wait a while' or 'wait a while and then transform'. To see what this definition implies we insert Eqs. (2) and (1) into Eq. (3)

$$\begin{aligned}
|\psi'(t)\rangle &= T|\psi(t)\rangle \\
e^{-iHt/\hbar}T|\psi(0)\rangle &= Te^{-iHt/\hbar}|\psi(0)\rangle \\
\left(e^{-iHt/\hbar}T - Te^{-iHt/\hbar}\right)|\psi(0)\rangle &= 0 \\
\left[e^{-iHt/\hbar}, T\right] &= 0
\end{aligned} \tag{4}$$

where in the last equality we have used that Eq. (3) must hold for all $|\psi(0)\rangle$. Expanding the exponential function and requiring that the Eq. (4) holds for any value of t it follows that $[H, T] = 0$. Thus the requirement T must fulfill in order to be a symmetry is

$$[H, T] = 0 \Rightarrow T^\dagger H T = H \tag{5}$$

($[H, T] = HT - TH = 0 \Rightarrow HT = TH \Rightarrow T^{-1}HT = H$ which for for unitary operators $T^\dagger = T^{-1}$ becomes $T^\dagger HT = H$.

Conserved quantities. The fact that $[H, T] = 0$ implies a conserved quantity. This is easiest to see by considering the Ehrenfest theorem for the time-change of the expectation value $\langle T\rangle$.

$$i\hbar\frac{d\langle T\rangle}{dt} = \langle [T, H]\rangle = 0. \tag{6}$$

[1]Adapted from J.J. Sakurai, Modern Quantum Mechanics, Addison-Wesley 1993

This means that $\langle T\rangle$ is a conserved quantity –it does not change in time. Another consequence is that eigenstates of T will remain eigenstates of T as time changes. This can be seen as follows. Let $|\chi(0)\rangle$ be an eigenstate of T at $t=0$ with eigenvalue g,

$$T|\chi(0)\rangle = g|\chi(0)\rangle. \tag{7}$$

The time-evolved state $|\chi(t)\rangle = e^{-iHt/\hbar}|\chi(0)\rangle$ will also be an eigenstate of T with the same eigenvalue g because

$$T|\chi(t)\rangle = Te^{-iHt/\hbar}|\chi(0)\rangle = e^{-iHt/\hbar}T|\chi(0)\rangle = ge^{-iHt/\hbar}|\chi(0)\rangle = g|\chi(t)\rangle. \tag{8}$$

Degeneracy. The existence of a symmetry implies also in certain cases the existence of degeneracies in the energy spectrum. Consider an energy eigenfunction $|n\rangle$ so that $H|n\rangle = E_n|n\rangle$. Then if T is a symmetry, $T|n\rangle$ is also an energy eigenfunction with the same energy because

$$H\left(T|n\rangle\right) = TH|n\rangle = TE_n|n\rangle = E_n\left(T|n\rangle\right) \tag{9}$$

The state $T|n\rangle$ does not need to be equal to the state $|n\rangle$. In the case where $T|n\rangle \neq |n\rangle$ the energy spectrum is degenerate. In fact one can turn this around, in most cases when there are degeneracies in the energy spectrum they are consequences of symmetries.

Examples:

Translational symmetry. Let us consider a state represented by a function $\psi(x)$. Then the action of the translation operator T_a which translates the function an amount a is

$$T_a\psi(x) = \psi(x+a). \tag{10}$$

We can express T_a as a collection of differential operators with the help of the Taylor-expansion

$$\psi(x+a) = \psi(x) + a\frac{d\psi}{dx} + \frac{a^2}{2!}\frac{d^2\psi}{dx^2} + \ldots = e^{a\frac{d}{dx}}\psi(x) = e^{iaP/\hbar}\psi(x) \tag{11}$$

where $P = (\hbar/i)d/dx$ is the momentum operator. Thus we can write

$$T_a = e^{iaP/\hbar}. \tag{12}$$

Therefore translations are intimately related to the momentum operator, we say that translations are *generated* by the momentum operator. If $[P,H]=0$ the Hamiltonian is translationally symmetric which implies the conservation of momentum. Similarly the generator of time-translations is the Hamiltonian which follows from the form $e^{-iHt/\hbar}$ of the time-development or time-translation operator. Thus time-translational invariance implies the conservation of energy.

Rotational symmetry. Let us consider the case of a rotation $R_z(\phi_a)$ around the z-axis by a finite amount ϕ_a. In analogy with the Taylor-expansion used in the case of translational symmetry this rotation operator can be written

$$R_z(\phi_a) = e^{i\phi_a L_z/\hbar} \tag{13}$$

where L_z is the angular momentum component in the z-direction which is represented by $L_z = (\hbar/i)d/d\phi$. Thus L_z generates rotations about the z-axis. Similarly rotations about the x and y-axis are generated by L_x and L_y. If R_z is a symmetry of the Hamiltonian, $[R_z, H] = 0$, it follows that $[L_z, H] = 0$ and that one can choose eigenfunctions for H that are simultaneously also eigenfunctions for L_z. Therefore we can label the eigenfunctions by the energy E_n and m, the eigenvalues of L_z. If the Hamiltonian is also rotationally symmetric about the x and y-axis then the action of R_x (or R_y) on a state with a definite m will yield states with $m+1$, m and $m-1$. This is because $L_x = (L_+ + L_-)/2$. Thus it will yield a state different from $|E_n, m\rangle$. Therefore the energy spectrum is degenerate. This is the reason why energy-levels in rotationally symmetric systems do not depend on the quantum number m.

Parity. The parity transformation Π is a space-inversion transformation about the origin. We define it in terms of its action on the position eigenkets

$$\Pi|\vec{r}\rangle = |-\vec{r}\rangle \tag{14}$$

Performing this transform twice results in the original state

$$\Pi^2|\vec{r}\rangle = \Pi|-\vec{r}\rangle = |\vec{r}\rangle \tag{15}$$

Thus $\Pi^2 = I$, the identity operator and therefore $\Pi = \Pi^{-1} = \Pi^\dagger$. It follows that the eigenvalues of Π are ± 1. Eigenstates of Π are referred to as being of even(odd) parity if they correspond to an eigenvalue +1(-1). It follows from Eq. (14) that

$$\langle\vec{r}|\Pi^\dagger \vec{r}\Pi|\vec{r}\rangle = \langle -\vec{r}|\vec{r}|-\vec{r}\rangle = -\langle\vec{r}|\vec{r}|\vec{r}\rangle \tag{16}$$

This should hold for any position eigenket, thus the following relation between operators must hold

$$\Pi^\dagger \vec{r}\Pi = -\vec{r} \Rightarrow \vec{r}\Pi = -\Pi\vec{r} \tag{17}$$

where we have used $\Pi^\dagger = \Pi$. A similar relation holds for the momentum operators

$$\Pi^\dagger \vec{p}\Pi = -\vec{p} \Rightarrow \vec{p}\Pi = -\Pi\vec{p} \tag{18}$$

One can use these relations to check whether or not $[\Pi, H] = 0$. Take the example of a free particle Hamiltonian with $H = \vec{p}^2/2m$. Then $[\Pi, H] \propto [\Pi, \vec{p}^2] = \Pi\vec{p}^2 - \vec{p}^2\Pi = -\vec{p}\Pi\vec{p} + \vec{p}\Pi\vec{p} = 0$. It follows that H is parity-symmetric and that Π and H are simultaneously diagonalizable. If the state of a given energy is unique (not degenerate), it will also be an eigenstate of the parity operator (either even or odd). The energy eigenfunctions of a free particle are proportional to plane waves: $e^{i\vec{k}\cdot\vec{r}}$ where $E = \hbar^2\vec{k}^2/2m$. These plane wave solutions are not

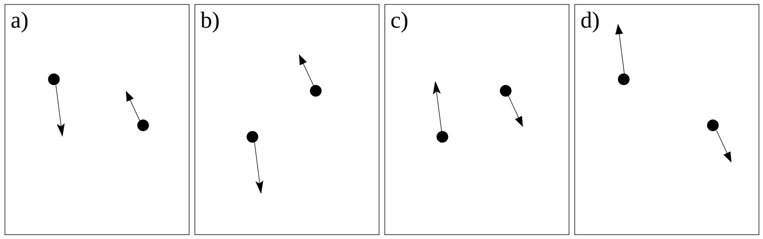

Figure 1: a) A state of two particles with their momenta shown as arrows. b) A time t later the two particles have moved while their momenta are unchanged. c) a time-reversal transformation is carried out, the momenta reverse their direction, then d) a time t after the reversal of the momenta the particles have returned to their original positions, as in a), but now with reversed momenta.

parity eigenstates, except the state with $\vec{k} = 0$ which is a parity even state. The reason is that there is degeneracy, free particle states with $\vec{k}$ have the same energy as states with $-\vec{k}$, thus the energy eigenstates are not necessarily parity eigenstates. However, it is possible to combine these energy eigenstates into parity eigenstates, $\cos(\vec{k} \cdot \vec{r})$ and $\sin(\vec{k} \cdot \vec{r})$ which are even respectively odd under parity.

Time-reversal. The time-reversal transformation is really a reversal of motion transformation. For the time-reversal transformation we must be careful about our definition of symmetry as it involves propagation in time. Let us denote the time-reversal transformation operator by Θ. Then if we have a state $|\psi(0)\rangle$ and evolve it in time we get

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle. \tag{19}$$

If we apply the time-reversal transformation to this time-evolved state and further evolve that state in time the same amount as we did first, we expect to get back the original state but with reversed motions, that is the time-reversed original state, see Fig. 1. Thus

$$e^{-iHt/\hbar}\Theta|\psi(t)\rangle = \Theta|\psi(0)\rangle \tag{20}$$

One can think of time-reversal as running a movie in reverse, where the time-reversal operator is the operation which reverses all directions of motion. This should hold for all states $|\psi(0)\rangle$, thus we find when inserting

$$e^{-iHt/\hbar}\Theta e^{-iHt/\hbar} = \Theta \tag{21}$$

or

$$e^{-iHt/\hbar}\Theta = \Theta e^{iHt/\hbar}. \tag{22}$$

Expanding $U(t) = 1 - iHt/\hbar + \ldots$ and requiring the equation to hold for all t we find

$$-iH\Theta = \Theta iH. \tag{23}$$

If Θ is a linear operator we can move the i on the right in front of ΘH and cancel the i's on both sides. However, this would mean that the time-reversed energy eigenstate $\Theta|E\rangle$ would be an energy eigenstate with negative energy $-E$. This can be seen from $H\Theta|E\rangle = -\Theta H|E\rangle = -E\Theta|E\rangle$. This cannot make sense as there cannot be a state with energy lower than the ground state. Therefore we come to the conclusion that Θ cannot be a linear operator. Instead we are dealing here with an antiunitary operator with the antilinear property $\Theta i = -i\Theta$. Using this property we get the condition $[H, \Theta] = 0$ for the system to be time-reversal symmetric. In some cases with time-reversal symmetry an energy eigenstate and its time-reversed version are different. This implies a two-fold degeneracy known as Kramers degeneracy.