FYS 3120/4120: Classical Mechanics and Electrodynamics

Formula Collection
Spring semester 2006

1 Analytical Mechanics

The Lagrangian

$$L = L(q, \dot{q}, t) \;, \tag{1}$$

is a function of the generalized coordinates $q = \{q_i \,;\, i = 1, 2, ..., D\}$ of the physical system and their time derivatives $\dot{q} = \{\dot{q}_i \,;\, i = 1, 2, ..., D\}$. The Lagrangian may also have an *explicit* dependence of time t.

Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \;, \quad i = 1, 2, .., D. \tag{2}$$

Generalized (canonical) momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \;, \quad i = 1, 2, .., D. \tag{3}$$

The Hamiltonian

$$H(p, q) = \sum_{i=1}^{D} \dot{q}_i p_i - L \tag{4}$$

is usually considered as a function of the generalized coordinates q and momenta p.

Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \;, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \;, \quad i = 1, 2, .., D \tag{5}$$

(6)

Standard expressions for L og H

$$\begin{aligned} L &= T - V \\ H &= T + V \end{aligned} \tag{7}$$

with T as kinetic energy and V as potential energy.

Charged particle in electromagnetic field (non-relativistic)

$$
\begin{aligned}
L = L(\mathbf{r},\mathbf{v}) &= \frac{1}{2}mv^2 - q\phi + q\mathbf{v}\cdot\mathbf{A} \\
H = H(\mathbf{r},\mathbf{p}) &= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi
\end{aligned}
\tag{8}
$$

2 Relativity

Space-time coordinates

$$
(x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{r})
\tag{9}
$$

General Lorentz transformation

$$
x^\mu \to x'^\mu = L^\mu_{\ \nu} x^\nu + a^\mu
\tag{10}
$$

Special Lorentz transformation with velocity v in the x direction

$$
\begin{aligned}
x'^0 &= \gamma(x^0 - \beta x^1) \\
x'^1 &= \gamma(x^1 - \beta x^0)
\end{aligned}
\tag{11}
$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, and x^2 og x^3 are unchanged.

Betingelse på Lorentztransformasjons-matrisene

$$
g_{\mu\nu} L^\mu_{\ \rho} L^\nu_{\ \sigma} = g_{\rho\sigma}
\tag{12}
$$

Invariant line element

$$
\Delta s^2 = \Delta\mathbf{r}^2 - c^2\Delta t^2 = g_{\mu\nu}\,\Delta x^\mu \Delta x^\nu = \Delta x_\mu\,\Delta x^\mu
\tag{13}
$$

Metric tensor

$$
g_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu \\ -1, & \mu = \nu = 0 \\ 1, & \mu = \nu \neq 0 \end{cases}
$$

Upper and lower index

$$
\begin{aligned}
&x_\mu = g_{\mu\nu}\, x^\nu, \quad (x^\mu) = (ct, \mathbf{r}), \quad (x_\mu) = (-ct, \mathbf{r}) \\
&x^\mu = g^{\mu\nu} x_\nu\,, \quad g_{\mu\rho}\, g^{\rho\nu} = \delta^\nu_\mu
\end{aligned}
\tag{14}
$$

Proper time - time dilatation

$$d\tau = \frac{1}{\gamma}dt, \quad \gamma = (1 - (\frac{v}{c})^2)^{-\frac{1}{2}} \tag{15}$$

$d\tau$: time interval measured in (instantanteous) rest frame (= proper time), dt: interval measured in arbitrarily chosen inertial system (coordinate time).

Length contraction

$$L = \frac{1}{\gamma}L_0 \tag{16}$$

L^0: length measured in rest frame, L: length measured (at simultaneity) in arbitrarily chosen inertial frame.

Four velocity

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma\,(c, \mathbf{v})\ , \qquad U^\mu U_\mu = -c^2 \tag{17}$$

Four acceleration

$$\mathcal{A}^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}\ , \quad \mathcal{A}^\mu U_\mu = 0 \tag{18}$$

Proper acceleration $\mathbf{a_0}$
Acceleration measured in instantaneous rest frame,

$$\mathcal{A}^\mu \mathcal{A}_\mu = \mathbf{a_0}^2 \tag{19}$$

Four momentum

$$p^\mu = m_0\, U^\mu = m_0\gamma(c, \mathbf{v}) = (\frac{E}{c}, \mathbf{p}) \tag{20}$$

with m_0 as rest mass.

Relativistic energy

$$E = \gamma m_0 c^2 \equiv mc^2 \tag{21}$$

3 Elektrodynamics

Maxwell's equations

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} &= \mu_0 \mathbf{j} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} &= 0
\end{aligned}
\tag{22}
$$

Maxwell's equations in covariant form

$$
\begin{aligned}
\partial_\nu F^{\mu\nu} &= \mu_0 j^\mu \ , \quad \partial_\nu \equiv \frac{\partial}{\partial x^\nu} \\
\partial_\nu \tilde{F}^{\mu\nu} &= 0 \ , \qquad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}
\end{aligned}
\tag{23}
$$

Electromagnetic field tensor

$$
\begin{aligned}
F^{0k} &= \frac{1}{c} E_k \ , \quad F^{ij} = \epsilon_{ijk} B_k \\
\tilde{F}^{0k} &= -B_k \ , \quad \tilde{F}^{ij} = \frac{1}{c}\epsilon_{ijk} E_k
\end{aligned}
\tag{24}
$$

Four-current density

$$
(j^\mu) = (c\rho, \mathbf{j}) \tag{25}
$$

Charge conservation

$$
\partial_\mu j^\mu = 0 \ , \quad \frac{\partial}{\partial t}\rho + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{26}
$$

Electromagnetic potentials

$$
\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A} \ , \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{27}
$$

Four potentials

$$
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ , \quad (A^\mu) = (\frac{1}{c}\phi, \mathbf{A}) \tag{28}
$$

Lorentz force
Force from the electromagnetic field on a charged point particle

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{29}$$

Potentials from charge and current distributions
In Lorentz gauge, $\partial_\mu A^\mu = 0$:

$$\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV' \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV' \end{aligned} \tag{30}$$

Retarded time

$$t' = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'| \tag{31}$$

Electric dipole moment

$$\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r}) dV \tag{32}$$

Electric dipole potential (dipole in origin)

$$\phi = \frac{\mathbf{n} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} \ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{33}$$

Force and torque (about the origin)

$$\mathbf{F} = \mathbf{p} \cdot \boldsymbol{\nabla}\mathbf{E} \ , \quad \mathbf{M} = \mathbf{p} \times \mathbf{E} \tag{34}$$

Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{j}(\mathbf{r}) \ dV \tag{35}$$

Magnetic dipole potential (dipole in the origin)

$$\mathbf{A} = \frac{\mu_0 \ \mathbf{m} \times \mathbf{n}}{4\pi r^2} \ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{36}$$

Force and torque (about the origin)

$$\mathbf{F} = \mathbf{m} \cdot \boldsymbol{\nabla}\mathbf{B} \ , \quad \mathbf{M} = \mathbf{m} \times \mathbf{B} \tag{37}$$

Lorentz transformation of the electromagnetic field

$$F'^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma F^{\rho\sigma} \tag{38}$$

Lorentz invariants

$$
\begin{aligned}
\mathbf{E}^2 - c^2\mathbf{B}^2 &= -\frac{c^2}{2} F_{\mu\nu}F^{\mu\nu} \\
\mathbf{E}\cdot\mathbf{B} &= \frac{c}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}
\end{aligned}
\tag{39}
$$

Special Lorentz transformations

$$
\begin{aligned}
&\mathbf{E}'_{||} = \mathbf{E}_{||}\ , \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}) \\
&\mathbf{B}'_{||} = \mathbf{B}_{||}\ , \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v}\times\mathbf{E}/c^2)
\end{aligned}
\tag{40}
$$

The fields are decomposed in a parallel component ($||$) along the direction of transformation and a perpendicular component ($\perp$), orthogonal to the direction of transformation.

Elektromagnetic field energy density

$$
u = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) = \frac{\epsilon_0}{2}(E^2 + c^2B^2) \tag{41}
$$

Elektromagnetic energy current density (Poyntings vektor)

$$
\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{42}
$$

Monochromatic plane waves, plane polarized

$$
\begin{aligned}
&\mathbf{E} = \mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)\ , \quad \mathbf{B} = \mathbf{B}_0\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) \\
&\mathbf{E}_0\cdot\mathbf{k} = \mathbf{B}_0\cdot\mathbf{k} = 0\ , \quad \mathbf{B}_0 = \frac{1}{c}\mathbf{n}\times\mathbf{E}_0\ , \quad \mathbf{n} = \frac{\mathbf{k}}{k}
\end{aligned}
\tag{43}
$$

Monochromatic plane waves, circular polarized

$$
\begin{aligned}
&\mathbf{E} = \mathbf{E}_0\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]\ , \quad \mathbf{B} = \mathbf{B}_0\exp[(\mathbf{k}\cdot\mathbf{r} - \omega t)] \quad \text{realdel} \\
&\mathbf{E}_0 = E_0\frac{1}{\sqrt{2}}(\mathbf{e}_1 + i\mathbf{e}_2)\ , \quad \mathbf{B}_0 = B_0\frac{1}{\sqrt{2}}(\mathbf{e}_2 - i\mathbf{e}_1)
\end{aligned}
\tag{44}
$$

Polarization vectors

$$
\mathbf{e}_1\cdot\mathbf{k} = \mathbf{e}_2\cdot\mathbf{k} = 0\ , \quad \mathbf{e}_1\cdot\mathbf{e}_2 = 0\ , \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1 \tag{45}
$$

Four-wave vector

$$
(k^\mu) = (\frac{\omega}{c}, \mathbf{k})\ , \quad \omega = ck \tag{46}
$$

Radiation, fields in the wave zone($r >> r', \lambda$)

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi c}\frac{\mathbf{n}}{r} \times \frac{d}{dt}\int \mathbf{j}(\mathbf{r}',t')dV' \ , \quad \mathbf{n} = \frac{\mathbf{r}}{r}$$
$$\mathbf{E}(\mathbf{r},t) = c\mathbf{B}(\mathbf{r},t) \times \mathbf{n} \qquad (47)$$

Elektric dipole radiation

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi c}\frac{\mathbf{n}}{r} \times \ddot{\mathbf{p}}_{(t-\frac{r}{c})} \ , \quad \mathbf{E}(\mathbf{r},t) = c\mathbf{B}(\mathbf{r},t) \times \mathbf{n} \qquad (48)$$

Radiation from accelerated charged particle

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 q}{4\pi c r}\mathbf{a}_{(t-\frac{r}{c})} \times \mathbf{n}$$
$$\mathbf{E}(\mathbf{r},t) = c\mathbf{B}(\mathbf{r},t) \times \mathbf{n} \ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \qquad (49)$$

Radiated power, Larmor's formula

$$P = \frac{\mu_0 q^2}{6\pi c}\mathbf{a}^2 \qquad (50)$$

Maxwell's equations in a medium

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_{fri} \\ \nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} &= \mathbf{j}_{fri} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} &= 0 \end{aligned} \qquad (51)$$

Polarization (LIH medium)

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(1+\chi)\mathbf{E} = \epsilon_r\epsilon_0\mathbf{E} = \epsilon\mathbf{E} \qquad (52)$$

Magnetization (LIH medium)

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu_r\mu_0\mathbf{H} = \mu\mathbf{H} \qquad (53)$$

Polarization charge (volume density)

$$\rho = \rho_{free} + \rho_p \ , \quad \rho_p = -\mathbf{\nabla} \cdot \mathbf{P} \tag{54}$$

Polarisation current and magnetization current (volume density)

$$\mathbf{j} = \mathbf{j}_{free} + \mathbf{j}_p + \mathbf{j}_m \ , \quad \mathbf{j}_p = \frac{\partial}{\partial t}\mathbf{P} \ , \quad \mathbf{j}_m = \mathbf{\nabla} \times \mathbf{M} \tag{55}$$