**FYS 3120 Classical Mechanics and Electrodynamics**
**Midterm Exam, Spring semester 2009**

**Return of solutions**
The problem set is available on the course page from Friday, March 20.
Deadline for return of solutions is Friday, March 27.
Return of solutions to *Ekspedisjonskontoret* in the Physics building.

**Language**
The solutions may be written in Norwegian or English.

**Questions**
concerning the text can be posed to Per Øyvind Sollid (office 469) or Jon Magne Leinaas (office 471).
We are available for questions on Friday March 20 and Monday, March 23.

**The problem set**
consists of 3 problems printed on 4 pages.

**PROBLEM 1**
**Particle in a periodic potential**

A particle of mass $m$ moves in a one-dimensional periodic potential

$$V(x) = V_0(\sin x + a\sin^2 x) \tag{1}$$

with $x$ as the coordinate in the direction of motion, $a$ as an external parameter that can be varied, and $V_0$ as a constant that measures the strength of the potential. We assume both $V_0$ and $a$ to be positive.

a) Determine the equilibrium points of the potential for different values of $a$, and indicate which of the equilibrium points that are stable and which ones are unstable. Discuss separately the cases $a < 1/2$ and $a > 1/2$.

b) Illustrate the situation by plotting the potential for the three values $a = 0$, $0.5$ and $1$. Discuss in what sense the situation changes when $a$ increases through the value $1/2$ and relate this to the results of point a).

c) Give the expression for the Lagrangian of the particle and use Lagrange's equation to find the equation of motion of the particle.

d) Assume the particle performs small oscillations about one of the stable equilibrium points, with coordinate denoted by $x_0$. We write the postion coordinate as $x = x_0 + \xi$ with $\xi << 1$. Show that this condition allows us to simplify the equation of motion so it takes the form of an harmonic oscillator equation for $\xi$. Determine the oscillation frequency as a function of $a$ for $a < 1/2$ and $a > 1/2$.

e) Find the Hamiltonian $H(x, p)$ of the system, with $p$ as the conjugate momentum of the coordinate $x$. Explain what is meant by considering $H(x, p)$ as a *phase space potential* and describe how the motion in the two-dimensional phase space is determined by this potential.

f) Make a contour plot of the phase space potential which show equipotential lines for the three different situations $a = 0$, $0.5$ and $1$. Indicate in the diagrams the direction of motion of the particle.

Discuss, based on the plots, what are the different types of motion of the particle and indicate in the diagrams the location of the limiting curves, called *separatrices*, that separate the different types of motion.

**PROBLEM 2**
**Fermat's principle**

Fermat's principle says that a light ray will follow the path between two points that has the minimal *optical length*, which is the path that takes the *shortest time* for light to propagate. In an optical medium the speed of light ($c_m$) is modified by the index of refraction $n$, so that

$$c_m = \frac{c}{n} \tag{2}$$

with $c$ as speed of light in vacuum. If the index is not a constant, but varies through the medium this implies that a light ray does not follow a straight line, but is bent.

a) Let us assume that the index of refraction depends on the vertical coordinate $y$, so that $n = n(y)$. A light ray is sent in the $x, y$-plane, with $x$ as a horizontal coordinate, between an initial point with coordinates $(x_1, y_1)$ and final point with coordinates $(x_2, y_2)$. Show that Fermat's principle means that the path of the light ray, $y(x)$, between these points gives a minimal value of action integral

$$S[y(x)] = \int_{x_1}^{x_2} n(y)\sqrt{1+y'^2}dx\ , \qquad y' = \frac{dy}{dx} \tag{3}$$

b) This minimization problem is equivalent to a Lagrange's equation. Formulate this equation and show that it leads to a differential equation for the path $y(x)$ of the light ray that can be written as

$$y'' = [\frac{d}{dy}\ln n(y)](1+y'^2)\ , \qquad y'' = \frac{d^2y}{dx^2} \tag{4}$$

c) Show that the second order differential equation (4) can be integrated to give the following first order equation

$$\left(\frac{n(y)}{n_0}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \tag{5}$$

with $n_0$ as a constant. We consider in the following the physical situation where a light ray is sent

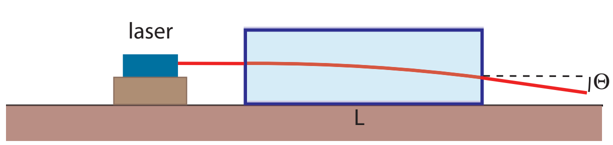


Figure 1: A laser beam is sent through a container with a sugar solutions. Due to variations in the strength of the solution the index of refraction decreases with height. This gives rise to a bending of the beam with $\theta$ as deflection angle.

through a container with a strong sugar solution. The container has the length $L$ in the $x$ direction. Due to the effect of gravity the strength of the solution varies with height, and this gives rise to a variable index of refraction of the form

$$n(y) = n_0\, e^{-\alpha y} \tag{6}$$

with $n_0$ and $\alpha$ as a constants.

d) Assume a light beam is sent in the horizontal direction ($x$ direction) into the container. The point of entering we give coordinates $x_1 = y_1 = 0$. Explain why Eq.(5) shows that the beam is deflected in the direction of *increasing* strength of the solution, that means here downwards.

e) Show that Eq.(5) i satisfied if we assume the following relation between $y$ and $x$ along the light path

$$e^{-\alpha y} = \frac{1}{\cos \alpha x} \tag{7}$$

f) The light beam leaves the container deflected by an angle $\theta$ relative to the incoming beam. Find an expression for the deflection angle in terms of $L$ and $\alpha$.

**PROBLEM 3**
**Particle in circular orbit**

An electron is circulating with constant speed in the accelerator ring LEP at CERN. The circumfer-

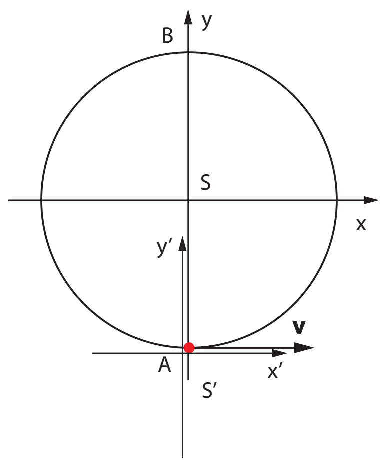


Figure 2: An electron circulates in an accelerator ring. $S$ denotes the laboratory frame where the ring is at rest and with the center of the ring as origin. $S'$ is the instantaneous rest frame of the electron, which is a t the origin of this reference frame when the electron is at the point $A$ of the ring.

ence of the ring is 27 km, and we assume it here to be completely circular with radius R. The speed of the electron corresponds to a gamma factor $\gamma = 10^5$.

The laboratory frame $S$ is the rest frame of the accelerator ring, and we assume that in the corresponding Cartesian coordinate frame the ring lies in the $x, y$ plane with the center of the ring at the origin. Since the electron is restricted to move in this plane we in the following simply neglect the $z$ coordinate and treat space-time as three-dimensional, with coordinates $(ct, x, y)$.

a) Determine the velocity $v$ of the electron in the lab frame $S$ relative to the speed of light. What is the period $T$ of circulation and the circular frequency $\omega$? Find the acceleration $a$ of the electron in the reference frame $S$.

b) Explain what is meant by the *proper time* $\tau$ of the particle and relate it to the coordinate time $t$ of the reference frame $S$ ? What is the period of circulation $T_\tau$ measured in proper time?

What is meant by the *proper acceleration* $a_0$ of the electron? Find the value of the electron's proper acceleration.

In the following we assume at an instant where the particle is located at the point $(x, y) = (0, -R)$ (point A in the figure) the time coordinates are $t = \tau = 0$.

c) Give the expressions for the coordinates of the electron's world line in $S$, $x^\mu(\tau)$, $\mu = 0, 1, 2$. (Express them as functions of proper time $\tau$, radius $R$ and circular frequency $\omega$.) Give the corresponding expressions for the components of the four-velocity $U^\mu(\tau)$ and the four-acceleration $A^\mu(\tau)$.

A second inertial frame $S'$ is introduced, which is the *instantaneous inertial rest frame* of the electron at time $t = 0$. The coordinate axes of $S$ and $S'$ are parallel and the electron is at the origin of Cartesian coordinate system of $S'$ at the instant $t' = 0$.

d) Explain what is meant by the instantaneous inertial rest frame, and give the transformation between the Cartesian coordinates of the two inertial frames $S$ and $S'$.

f) At time $t' = 0$ the accelerator ring defines a deformed circle in $S'$. Show that it is an ellipse and determine the lengths of the long and short axes.

g) What are the coordinates $x'^\mu(\tau)$, $\mu = 0, 1, 2$ of the electron's world line in reference system $S'$? Make a graphical representation of the trajectory in the $x', y'$ plane. (A different scale for the two directions of the plane may be used.)

h) In reference frame $S'$ the magnitude of the acceleration $a'$ of the electron will change with its position in the accelerator ring. Where on the ring will it have its maximum value? Explain your answer - detailed calculation is not needed.