FYS 3120 Classical Mechanics and Electrodynamics
Spring semester 2009

Problem Set 7

Problem 7.1 (Midterm Exam 2007)

According to Fermat's Principle, a light ray will follow the path between two points which makes the *optical path length* extremal. For simplicity we consider here paths constrained to a two dimensional plane (the x, y plane), in an optical medium with a position dependent index of refraction $n(x, y)$. The optical path length between two points (x_1, y_1) and (x_2, y_2) along $y(x)$ can be written as the integral

$$A[y(x)] = \int_{x_1}^{x_2} n(x,y)\sqrt{1+y'^2}dx \ , \qquad y' = \frac{dy}{dx} \tag{1}$$

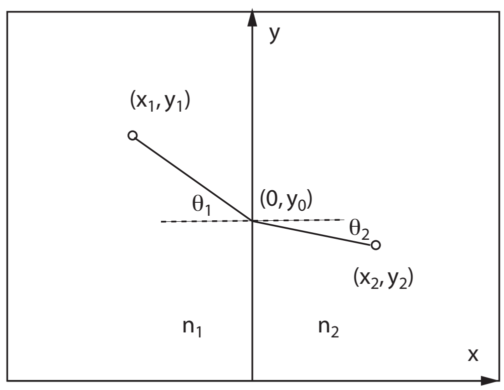

Figure 1:

a) Find Lagrange's equation for the variational problem $\delta A = 0$, and express it as a differential equation for the function $y(x)$. Show that if the index of refraction is constant the equation has the straight line between the two points as solution.

b) Assume the medium to have two different, constant indices of refraction, $n = n_1$ for $x < 0$ og $n = n_2$ for $x > 0$ (ses Fig. 1). Explain why the variational problem can now be simplified to the problem of finding the coordinate $y = y_0$ for the point where the light ray crosses the boundary between the two media at $x = 0$. Find the equation for y_0 that gives the shortest optical path length. (Solving the equation is not needed.)

c) Show that the equation for y_0 at point b) implies that the path of the light ray satisfies Snell's law of refraction,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{2}$$

with θ_1 and θ_2 as the angle of the light ray relative to the normal on the two sides of the the boundary.

Problem 7.2 (Midterm Exam 2006)

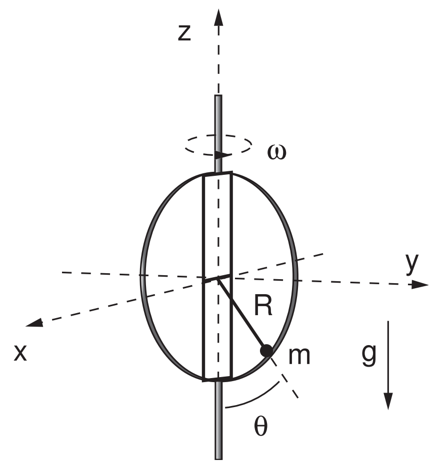

Figure 2:

A circular hoop is rotating with constant angular velocity ω around a symmetry axis with vertical orientation, as shown in Fig. 1. Inside the hoop a planar pendulum can perform free oscillations, while the plane of the pendulum rotates with the hoop. The mass of the pendulum bob is m, the length of the pendulum rod is R and the gravitational acceleration is g. The pendulum rod is considered as massless. As generalized coordinate we use the angle θ of the pendulum relative to the vertical axis.

a) Express the Cartesian coordinates of the pendulum bob as functions of θ and ω and find the Lagrangian of the pendulum.

b) Derive Lagrange's equation for the system. Find the oscillation frequency for small oscillations about the equilibrium point $\theta = 0$

c) Show that $\theta = 0$ is a *stable* equilibrium only for $\omega < \omega_{cr}$ and determine ω_{cr}. Show that for $\omega > \omega_{cr}$ there are two new equilibria $\theta_{\pm} \neq 0, \pi$ and determine the values of θ_+ and θ_- as functions of ω.

d) Study small deviations from equilibrium, $\theta = \theta_{\pm} + \chi$, with $\chi << 1$. Show that, for $\omega > \omega_{cr}$, the system will perform harmonic oscillations about the points θ_+ and θ_-. What are the corresponding oscillation frequencies?

The phenomenon where the original stable equilibrium $\theta = 0$ splits into two new equilibrium points θ_+ and θ_- is referred to as a *bifurcation*.

e) Find the Hamiltonian H of the system as function of θ and its conjugate momentum p_θ and derive the corresponding Hamilton's equations.

f) Consider the Hamiltonian $H(\theta, p_\theta)$ as a potential function of the two phase space variables θ and p_θ. Make a sketch of the equipotential lines $H(\theta, p_\theta) = const$ for the region around the equilibrium point $\theta = p_\theta = 0$, first in the case $\omega < \omega_{cr}$, and next in the case ω slightly larger than ω_{cr} (include in this case the new equilibrium points $(\theta_{\pm}, p_\theta = 0)$ in the drawing). Indicate in the drawing the direction of motion in the two-dimensional phase space. (A qualitative drawing is sufficient.)