

Exam FYS3120, 2014

Solutions

Problem 1

Lagrangian,

$$\begin{aligned} \text{a) } L &= \frac{1}{2} m \vec{v}^2 - V(\vec{r}) + q \vec{v} \cdot \vec{A} \\ &= \frac{1}{2} m \dot{\vec{r}}^2 - \frac{1}{2} m \omega_0^2 r^2 - \frac{1}{2} q \vec{v} \cdot (\vec{r} \times \vec{B}) \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{1}{2} m \omega_0^2 r^2 + \frac{1}{2} q B (\vec{r} \times \vec{v})_z \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 (\dot{\phi}^2 + \omega_B \dot{\phi} - \omega_0^2)) \end{aligned} \quad \begin{array}{l} = r^2 \dot{\phi} \\ \omega_B = \frac{qB}{m} \end{array}$$

$$\text{b) } \phi \text{ cyclic: } \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow p_\phi = \frac{\partial L}{\partial \dot{\phi}} = l \text{ constant}$$

$$l = m r^2 (\dot{\phi} + \frac{1}{2} \omega_B) \Rightarrow \dot{\phi} = \frac{l}{m r^2} - \frac{1}{2} \omega_B$$

Physical interpretation: angular momentum,
mechanical + electromagnetic contributions

Second const. of motion:

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0$$

$$H = p_r \dot{r} + p_\phi \dot{\phi} - L = \frac{1}{2} m (\dot{r}^2 + r^2 (\dot{\phi}^2 + \omega_0^2)) = T + V$$

energy conserved

$$c) \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial L}{\partial r} = m r (\dot{\varphi}^2 + \omega_B \dot{\varphi} - \omega_0^2)$$

Lagrange's equation \Rightarrow

$$\ddot{r} - r (\dot{\varphi}^2 + \omega_B \dot{\varphi} - \omega_0^2) = 0$$

$$l^2 = m^2 r^4 (\dot{\varphi}^2 + \omega_B \dot{\varphi} + \frac{1}{4} \omega_B^2)$$

$$\Rightarrow \ddot{r} - \frac{l^2}{m^2 r^3} + r (\omega_0^2 + \frac{1}{4} \omega_B^2) = 0$$

d) Circular motion: $r = \text{const} \Rightarrow \ddot{r} = 0$

$$\Rightarrow r^4 = \frac{l^2}{m^2 \Omega^2} \quad \Omega \equiv \sqrt{\omega_0^2 + \frac{1}{4} \omega_B^2}$$

$$\Rightarrow \underline{r = \sqrt{\frac{l}{m\Omega}}} \quad \text{radius of circle as function of } l \ (l \neq 0)$$

Angular velocity

$$\dot{\varphi} = \frac{l}{m r^2} - \frac{1}{2} \omega_B = \Omega - \frac{1}{2} \omega_B = \underline{\sqrt{\omega_0^2 + \frac{1}{4} \omega_B^2} - \frac{1}{2} \omega_B}$$

If $l = 0$:

$$\ddot{r} + \Omega^2 r = 0 \quad \text{harmonic osc. eq.}$$

$$\Rightarrow \underline{r = r_0 \cos \Omega t} \quad r_0 \text{ amplitude of radial oscillations}$$

Angle φ of direction of oscillations

$$\dot{\varphi} = -\frac{1}{2} \omega_B \Rightarrow \underline{\varphi = \varphi_0 - \frac{1}{2} \omega_B t}$$

rotation with ang. frequency $-\frac{1}{2} \omega_B$

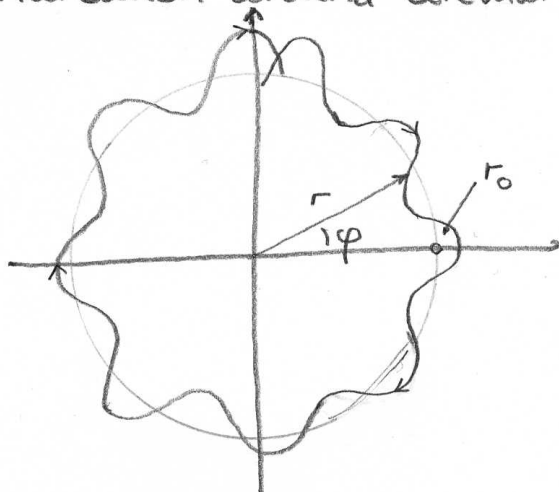
φ_0, r_0 determined by initial conditions

Typical motion ($l \neq 0$):

Radial oscillations about equilibrium value $r_0 = \sqrt{\frac{l}{m\Omega}}$,

combined with angular velocity $\dot{\phi} = \frac{l}{mr^2} - \frac{1}{2}\omega_B$

\Rightarrow perturbation around circular motion



Problem 2

a) Eq. of motion

$$\dot{\vec{p}} = e\vec{v} \times \vec{B} ; \quad \vec{p} = \gamma m \vec{v}$$

$$\dot{\vec{p}} \cdot \vec{v} = 0 \Rightarrow v, \gamma \text{ constant}$$

$$\gamma m \vec{a} = e\vec{v} \times \vec{B} \Rightarrow \gamma m a = e v B$$

\vec{B} const. \Rightarrow circular motion with const. angular velocity

$$v = \omega R, \quad a = \omega^2 R = \frac{v^2}{R}$$

This gives $\gamma m \frac{v^2}{R} = e v B \Rightarrow e B = \frac{\gamma m v}{R} = \underline{\underline{\frac{p}{R}}}$

b) Energy

$$E^2 = p^2 c^2 + m^2 c^4 = \gamma^2 m^2 c^4$$

$$\Rightarrow \gamma^2 = \frac{p^2}{m^2 c^2} + 1$$

$$\gamma = \sqrt{\frac{p^2}{m^2 c^2} + 1} \approx \frac{p}{m c} = \frac{7 \cdot 10^{12}}{938 \cdot 10^6} = \underline{\underline{7460}}$$

Note: $\gamma \gg 1 \Rightarrow 1 - \beta^2 \ll 1$ v very close to c .

Acceleration in lab frame

$$a = \frac{v^2}{R} \approx \frac{c^2}{R} = \frac{9 \cdot 10^8}{2804} \text{ m/s}^2 = \underline{\underline{3.2 \cdot 10^{13} \text{ m/s}^2}}$$

c) Transformation of fields to instantaneous rest frame:

Since $\vec{v} \cdot \vec{B} = 0$, $\vec{E} = 0$ in lab frame, the general formulas are simplified to

$$\vec{B}' = \gamma \vec{B}, \quad \vec{E}' = \gamma \vec{v} \times \vec{B}$$

Field strengths

$$B' = \gamma B = 7460 \times 8.33 \text{ T} = \underline{62164 \text{ T}}$$

orientation: orthogonal to the plane of motion

$$E' = \gamma v B \approx v B' = 3.0 \cdot 10^8 \cdot 62164 \text{ T m/s} = \underline{1.86 \cdot 10^{13} \text{ V/m}}$$

orientation: in the plane of motion, orthogonal to the orbit

Proper acceleration = acceleration in the instantaneous rest frame

$$m \vec{a}_0 = e \vec{E}' \quad (v' = 0 \Rightarrow \gamma' = 1)$$

$$= \gamma e \vec{v} \times \vec{B}$$

Compare with acceleration in lab frame

$$\gamma m \vec{a} = e \vec{v} \times \vec{B}$$

$$\Rightarrow \vec{a}_0 = \gamma^2 \vec{a}, \quad a_0 = 7460^2 \cdot 3.2 \cdot 10^{13} \text{ m/s}^2 = \underline{1.79 \cdot 10^{21} \text{ m/s}^2}$$

Problem 3

a) Electric dipole moment

$$\begin{aligned}\vec{p} &= \int_{-a/2}^{a/2} z \lambda(z,t) dz \vec{k} \\ \Rightarrow \dot{\vec{p}} &= \int_{-a/2}^{a/2} z \frac{\partial \lambda}{\partial t} dz \vec{k} \\ &= - \int_{-a/2}^{a/2} z \frac{\partial I_1}{\partial z} dz \vec{k} \\ &= \int_{-a/2}^{a/2} I_1(z,t) dz \vec{k} - \left[z I_1 \right]_{-a/2}^{a/2} \vec{k} \\ &= \frac{a}{\pi} I_0 \sin \omega t \left[\sin \frac{\pi z}{a} \right]_{-a/2}^{a/2} \vec{k} = \underline{\underline{\frac{2a}{\pi} I_0 \sin \omega t \vec{k}}}\end{aligned}$$

Magnetic dipole moment

$$\begin{aligned}\vec{m} &= \frac{1}{2} \oint \vec{r} \times I_2 d\vec{r} = \frac{1}{2} I_0 \sin \omega t (2a)^2 \int_0^{2\pi} d\phi \vec{k} \\ &= \underline{\underline{4\pi a^2 I_0 \sin \omega t \vec{k}}}\end{aligned}$$

$$\Rightarrow \ddot{\vec{p}} = \frac{2a}{\pi} \omega I_0 \cos \omega t \vec{k}, \quad \ddot{\vec{m}} = -4\pi a^2 \omega^2 I_0 \sin \omega t \vec{k}$$

b) Define $\vec{k} \times \vec{n} = \sin \theta \vec{e}_1$, $(\vec{k} \times \vec{n}) \times \vec{n} = \sin \theta \vec{e}_2$
 $\Rightarrow \vec{e}_1 \cdot \vec{e}_2 = 0$, $\vec{e}_1 \cdot \vec{n} = \vec{e}_2 \cdot \vec{n} = 0$, $\vec{e}_1^2 = \vec{e}_2^2 = 1$

Poynting's vector

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} \vec{E}^2 \vec{n} \\ &= \frac{\mu_0}{16\pi^2 r^2 c} \left(\ddot{p}^2 \sin^2 \theta \vec{e}_1^2 + \frac{1}{c^2} \ddot{m}^2 \sin^2 \theta \vec{e}_2^2 - \frac{2}{c} \ddot{p} \cdot \ddot{m} \sin^2 \theta \vec{e}_1 \cdot \vec{e}_2 \right) \vec{n} \\ &= \frac{\mu_0}{16\pi^2 r^2 c} \left(\ddot{p}^2 + \frac{1}{c^2} \ddot{m}^2 \right) \sin^2 \theta\end{aligned}$$

Equal time averages

$$\overline{\dot{\vec{p}}^2} = \frac{1}{c^2} \overline{\dot{\vec{m}}^2} \Rightarrow \frac{2}{\pi^2} (a\omega)^2 I_0^2 = 8\pi^2 \frac{(a\omega)^4}{c^2} I_0^2$$

$$\Rightarrow a\omega = \frac{c}{2\pi^2}$$

Poynting's vector

$$\vec{S} = \frac{\mu_0}{16\pi^2 r^2 c} \frac{4}{\pi^2} \left(\frac{c}{2\pi^2}\right)^2 I_0^2 \sin^2\theta \vec{n}$$

$$= \frac{\mu_0}{16\pi^8 r^2} c I_0^2 \sin^2\theta \vec{n}$$

Power per unit solid angle in direction \vec{n}

$$\frac{dP}{d\Omega} = r^2 \vec{S} \cdot \vec{n} = \frac{\mu_0 c}{16\pi^8} I_0^2 \sin^2\theta \quad \text{time independent}$$

c) Electric field

$$\vec{E}(\vec{r}, t) = \frac{\mu_0 c}{4\pi^4 r} I_0 (\sin\omega t \vec{e}_1 + \cos\omega t \vec{e}_2)$$

\vec{E} rotates in the plane spanned by \vec{e}_1 and \vec{e}_2 :

Circular polarization, lefthanded (clockwise around \vec{n})

If $a\omega \neq \frac{c}{2\pi^2}$: Different coefficients before $\sin\omega t \vec{e}_1$ and $\cos\omega t \vec{e}_2 \Rightarrow$ elliptic polarization with \vec{e}_1 and \vec{e}_2 def. symmetry axes

d) Circular receiver antenna: Current induced by e.m.f.,

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_m}{dt} \quad (\Phi_m \text{ magnetic flux through the loop})$$

Max oscillation amplitude of Φ_m , when the antenna oriented with \vec{n} in the plane of the antenna.