Midterm Exam FYS 3120, 2014

Solutions

Problem 1

a) Cartesian coordinates

$$x = R\sin\theta\cos\omega t$$
$$y = R\sin\theta\sin\omega t$$
$$z = -R\cos\theta$$

Velocities

$$\dot{x} = R(\dot{\theta}\cos\theta\cos\omega t - \omega\sin\theta\sin\omega t)$$
$$\dot{y} = R(\dot{\theta}\cos\theta\sin\omega t + \omega\sin\theta\cos\omega t)$$
$$\dot{z} = R\dot{\theta}\sin\theta$$

Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - mgz = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + mgR\cos\theta$$

$$= \frac{1}{2}mR^2\dot{\theta}^2 - W(\theta) \quad \underline{\text{with } W(\theta) = -(mgR\cos\theta + \tfrac{1}{2}mR^2\omega^2\sin^2\theta)}$$

Additional term is a sentrifugal term

b) $\frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta}$, $\frac{\partial L}{\partial\theta} = -\frac{dW}{d\theta} = -mgR\sin\theta + \frac{1}{2}mR^2\omega^2\sin 2\theta$

Lagrange's equation : $mR^2\ddot{\theta} + \frac{dW}{d\theta} = 0$

$$\Rightarrow \quad \underline{\ddot{\theta} + \frac{g}{R}\sin\theta - \frac{1}{2}\omega^2\sin 2\theta = 0}$$

small oscillations about $\theta = 0$: $\sin\theta \simeq \theta$, $\sin 2\theta \simeq 2\theta$

$$\underline{\ddot{\theta} + \left(\frac{g}{R} - \omega^2\right)\theta = 0} \quad \text{Angular freq. } \underline{\Omega = \sqrt{\frac{g}{R} - \omega^2}}$$

c) Condition for stable equilibrium, $\frac{d^2W}{d\theta^2}(\theta=0) > 0$

This is equivalent to $\frac{g}{R} - \omega^2 > 0$, which is satisfied when $\underline{\omega < \sqrt{\frac{g}{R}} \equiv \omega_{cr}}$

Equilibrium points, general condition

$$\frac{dW}{d\theta} = 0 \Rightarrow mgR\sin\theta - mR^2\omega^2\sin\theta\cos\theta = 0$$

Two types solution

1) $\sin\theta = 0 \Rightarrow \theta = 0, \pi$

2) $\cos\theta = \frac{g}{R\omega^2} \Rightarrow \underline{\theta = \pm \operatorname{Arccos}\left(\frac{g}{R\omega^2}\right) \equiv \theta_\pm}$

Solution 2) valid only when $\frac{g}{R\omega} < 1 \Rightarrow \underline{\omega > \omega_{cr}}$

d) Stability of equilibrium points $\theta_\pm$

$$\frac{d^2W}{d\theta^2} = mgR\cos\theta - mR^2\omega^2(2\cos^2\theta - 1)$$

$$\Rightarrow \frac{d^2W}{d\theta^2}(\theta=\theta_\pm) = mgR\frac{g}{R\omega^2} - mR^2\omega^2\left(2\frac{g^2}{R^2\omega^4} - 1\right)$$

$$= \underline{mR^2\omega^2\left(1 - \frac{\omega_{cr}^4}{\omega^4}\right)} > 0 \quad \text{when } \omega > \omega_{cr}$$

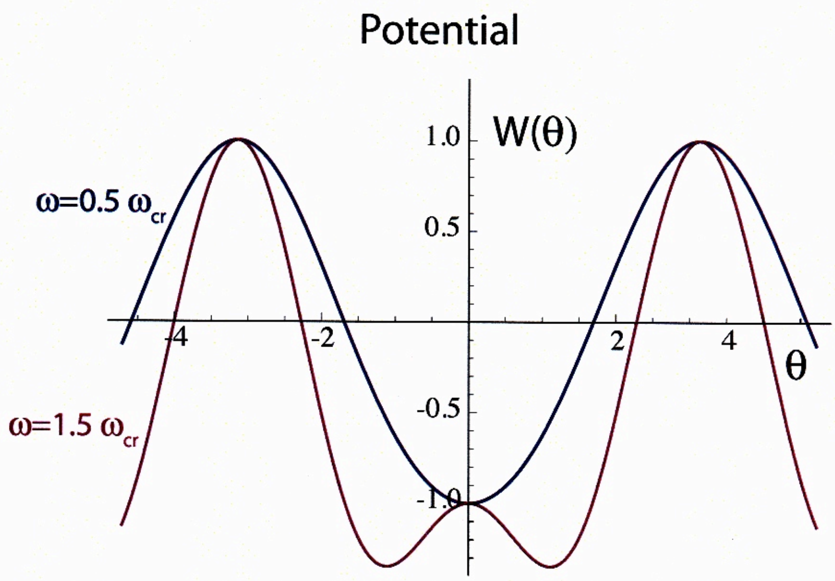
Proves stable equilibrium at $\theta = \theta_\pm$ — " —

Plots of $W(\theta)$ for $\omega = 0.5\,\omega_{cr}$ and $\omega = 1.5\,\omega_{cr}$ shown on separate sheet

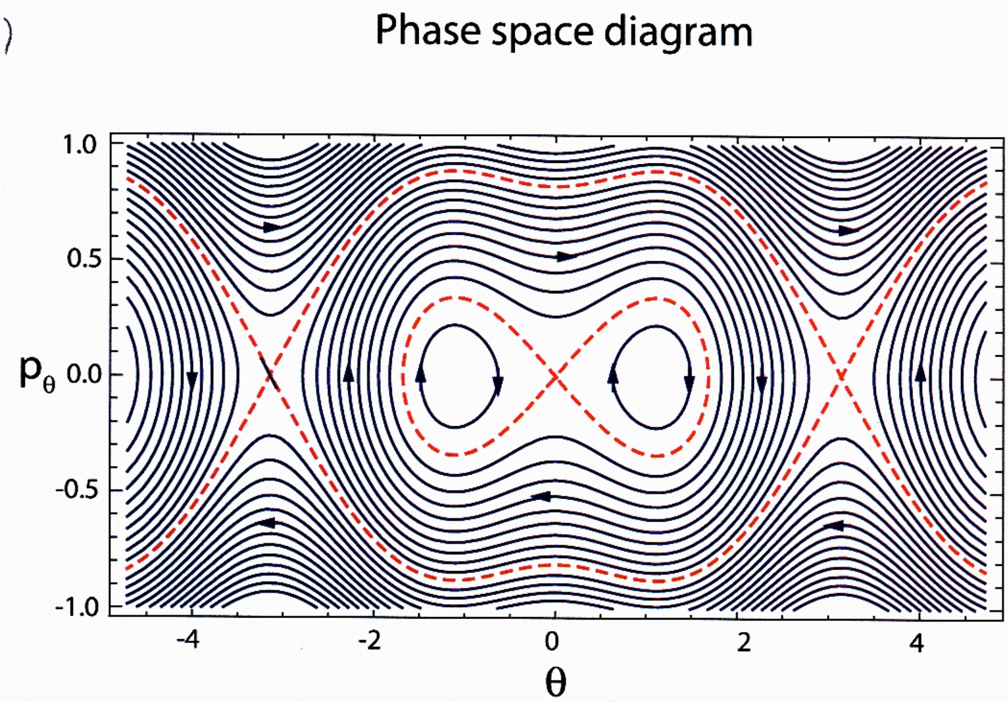
e) Hamiltonian $H = p_\theta\dot\theta - L$, $p_\theta = \frac{\partial L}{\partial\dot\theta} = mR^2\dot\theta$

$$\Rightarrow H = \frac{p_\theta^2}{2mR} + W(\theta) = \underline{\frac{p_\theta^2}{2mR} - mgR\cos\theta - \frac{1}{2}mR^2\omega^2\sin^2\theta}$$

No explicit time dependence: $\frac{\partial H}{\partial t} = 0 \Rightarrow \underline{\frac{dH}{dt} = 0}$ const. of motion d)



f)



The motion along equipotential lines (H const.), direction determined by gradient of H (Hamilton's eq.)

Three types of motion, 1) Circulation (oscillation) about one minimum, 2) circulation (oscillation) about two minima, 3) unbounded (non-oscillatory) motion. Separatrices shown in red.

Problem 2

a) Total time measured on earth

$$T_e = \frac{2D}{v} = \frac{10}{3}\frac{D}{c} = \underline{100 \text{ days}}$$

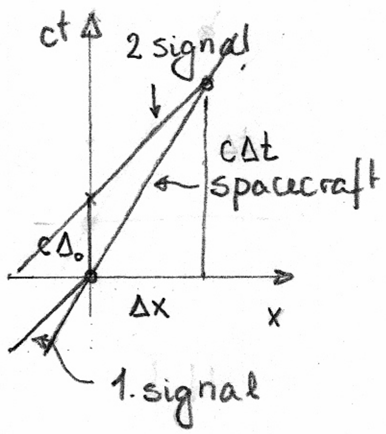
Total time measured on spacecraft, equal to proper time of spacecraft

General formula $d\tau = \frac{1}{\gamma} dt$ $\quad$ τ = proper time, t = coordinate time (on earth)

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\frac{9}{25}}} = \frac{5}{4}$$

$$\Rightarrow T_s = \frac{1}{\gamma} T_e = \frac{4}{5} T_e = \underline{80 \text{ days}}$$

b) Consider two subsequent signals, on the way out



Coordinate differences between events when signals are received

$$\Delta x = v \Delta t = c(\Delta t - \Delta_0)$$

$$\Rightarrow \quad \Delta t = \frac{c}{c-v} \Delta_0 = \frac{5}{2} \Delta_0$$

Time difference measured on spacecraft

$$\Delta_1 = \frac{1}{\gamma} \Delta t = \frac{4}{5} \cdot \frac{5}{2} \Delta_0 = \underline{2\Delta_0 = 2 \text{hours}}$$

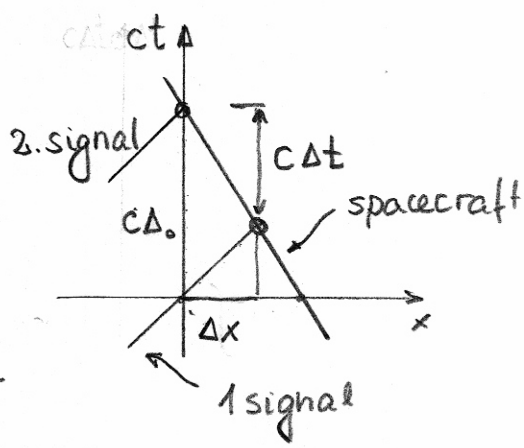
On the way back

$$\Delta x = v \Delta t = c(\Delta_0 - \Delta t)$$

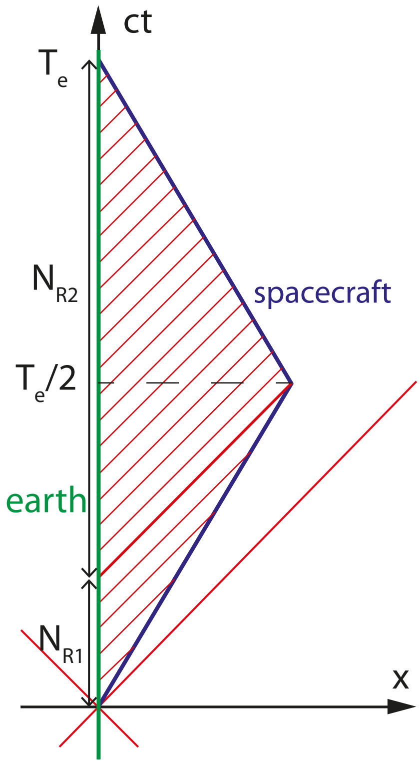
$$\Rightarrow \Delta t = \frac{c}{c+v} \Delta_0 = \frac{5}{8} \Delta_0$$

Registered on space craft

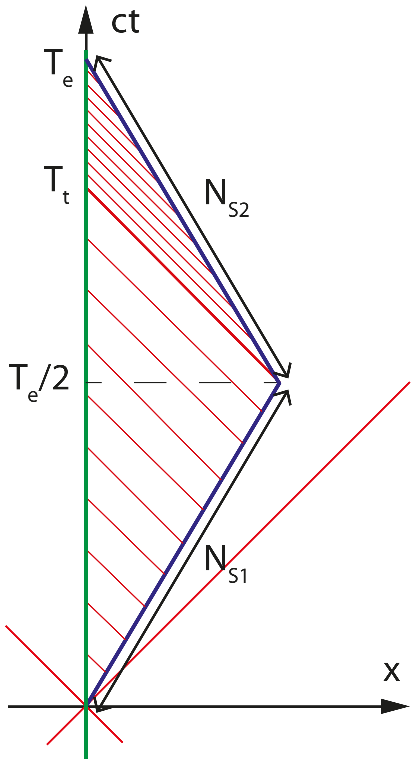
$$\Delta_2 = \frac{1}{\gamma} \Delta t = \frac{4}{5} \cdot \frac{5}{8} \Delta_0 = \underline{\frac{1}{2} \Delta_0 = 30 \text{min}}$$



Minkowski diagrams



signals sent from earth



signals sent from the spacecraft

c) The situation with regards to the time intervals of the signals at the receiver is fully symmetric wether the signals are sent from the space craft or from earth. Only the relative velocity matters. That is seen by the expressions

$$\frac{\Delta_1}{\Delta_0} = \frac{1}{\gamma}\frac{1}{1-\beta} = \sqrt{\frac{1+\beta}{1-\beta}}\,, \qquad \frac{\Delta_2}{\Delta_0} = \frac{1}{\gamma}\frac{1}{1+\beta} = \sqrt{\frac{1-\beta}{1+\beta}}$$

The expressions are the same as in the Doppler effect, in the first case with the receiver moving away from the sender and in the second case moving towards the sender

The time when the received signals change in intervals at earth is identical to the time when the signal sent at half time at the spacecraft reaches the earth,

$$T_t = \frac{1}{2}T_e + \frac{D}{c} = \frac{D}{v} + \frac{D}{c} = \frac{D}{c}\left(\frac{1}{\beta}+1\right) = \frac{8}{3}\cdot\frac{D}{c} = \underline{80\,\text{days}}$$

d) Number of signals received on the space craft, on the way out

$$N_{R1} = \frac{T_s/2}{\Delta_1} = \frac{1}{4}\frac{T_s}{\Delta_0} = \frac{1}{4}80\cdot 24 = \underline{480}$$

on the way back

$$N_{R2} = \frac{T_s/2}{\Delta_2} = \frac{T_s}{\Delta_0} = 80\cdot 24 = \underline{1920}$$

Number of signals sent, is the same out and back

$$N_{S1} = N_{S2} = \frac{T_s/2}{\Delta_0} = \underline{960}$$

Total numbers

$$N_R = N_{R1} + N_{R2} = \underline{2400}$$

$$N_S = N_{S1} + N_{S2} = \underline{1920}$$

N_R is the total number of signals sent from earth, at intervals Δ_0

$$\Rightarrow \quad N_R = \frac{T_e}{\Delta_0} = 100 \times 24 = 2400$$

N_S is the total number sent from the spacecraft, at intervals Δ_0

$$N_S = \frac{T_S}{\Delta_0} = 80 \times 24 = 1920$$

Ratio $\quad \frac{N_R}{N_S} = \frac{T_e}{T_S} = \gamma = \frac{5}{4}$

$$= \underline{\frac{2400}{1920}}$$ consistent results

Problem 3

a) Optical path length

$$A[y(x)] \equiv \int_{x_1}^{x_2} L(y,y',x)\,dx$$

$$L(y,y',x) = n(x,y)\sqrt{1+y'^2}\,dx$$

Variational problem: $\delta A = 0$ for variations with fixed end points (x_1,y_1) and (x_2,y_2) is equivalent to Lagrange's equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow \underline{\frac{d}{dx}\left(\frac{n(x,y)\,y'}{\sqrt{1+y'^2}}\right) - \frac{\partial n}{\partial y}\sqrt{1+y'^2} = 0}$$

$n = \text{constant} \Rightarrow \quad \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \text{constant} \Rightarrow y' = k \text{ (const.)}$$

$$\Rightarrow \underline{y(x) = kx + y_0} \quad \text{straight line}$$

b) Medium with two values of n

$$n = n_1 \text{ for } x<0, \quad n = n_2 \text{ for } x>0$$

Optical path a straight line on each side of $x=0$. For paths which cross the interface at $x=0$, the straight lines meet at a point $y=y_0$, and A becomes a function of y_0.

$$A = n_1\sqrt{x_1^2+(y_1-y_0)^2} + n_2\sqrt{x_2^2+(y_2-y_0)^2}$$

The variational problem is then reduced to

$$\frac{dA}{dy_0} = 0 \implies \underline{n_1 \frac{y_0-y_1}{\sqrt{x_1^2+(y_0-y_1)^2}} + n_2 \frac{y_0-y_2}{\sqrt{x_2^2+(y_0-y_1)^2}} = 0}$$

c) Relation to angles θ_1 and θ_2

$$\sin\theta_1 = \frac{y_1-y_0}{\sqrt{x_1^2+(y_0-y_1)^2}}, \quad \sin\theta_2 = \frac{y_0-y_2}{\sqrt{x_2^2+(y_0-y_0)^2}}$$

$$\Rightarrow \quad \underline{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$