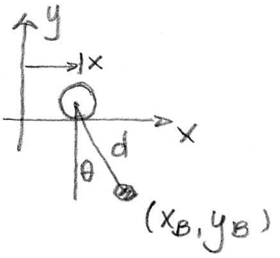
Exam Fys 3120, 2015

Solutions

Problem 1



a) $x_B = x + d\sin\theta$

$y_B = -d\cos\theta$

$\dot{x}_B = \dot{x} + d\cos\theta\,\dot{\theta}\,,\quad \dot{y}_B = d\sin\theta\,\dot{\theta}$

Rolling cylinder: $\dot{x} = \omega R\,,\ I = \frac{1}{2}mR^2 \Rightarrow \frac{1}{2}I\omega^2 = \frac{1}{4}m\dot{x}^2$

Kinetic energy

$$T_A = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{3}{4}m\dot{x}^2$$

$$T_B = \frac{1}{2}m(\dot{x}_B^2 + \dot{y}_B^2) = \frac{1}{2}m(\dot{x}^2 + d^2\dot{\theta}^2 + 2d\dot{x}\dot{\theta}\cos\theta)$$

Potential energy

$$V = mg\,y_B = -mgd\cos\theta$$

Lagrangian

$$L = T - V = \underline{m\left(\frac{5}{4}\dot{x}^2 + \frac{1}{2}d^2\dot{\theta}^2 + d\cos\theta\,\dot{x}\dot{\theta} + gd\cos\theta\right)}$$

Constants of motion

$$\frac{\partial L}{\partial x} = 0 \text{ (x cyclic)} \Rightarrow p_x = \frac{\partial L}{\partial \dot{\theta}} = m\left(\frac{5}{2}\dot{x} + d\cos\theta\,\dot{\theta}\right) \text{ constant}$$

initial condition $\Rightarrow p_x = 0 \Rightarrow \underline{\dot{x} = -\frac{2}{5}d\cos\theta\,\dot{\theta}}$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0 \text{ (energy conserved)}$$

$$H = \dot{x}p_x + \dot{\theta}p_\theta - L = m\left(\frac{5}{4}\dot{x}^2 + \frac{1}{2}d^2\dot{\theta}^2 + d\cos\theta\,\dot{x}\dot{\theta} - gd\cos\theta\right) = T + V$$

initial condition $H = E = \underline{-mgd\cos\theta_0}$

b) Lagrange's equations

1) $\frac{\partial L}{\partial \dot{x}} = \text{const} \Rightarrow p_x = 0 \qquad \underline{\dot{x} = -\frac{2}{5} d\cos\theta\,\dot{\theta}}$

2) $\frac{\partial L}{\partial \dot{\theta}} = m(d^2\dot{\theta} + d\cos\theta\,\dot{x}) = md^2(1-\frac{2}{5}\cos^2\theta)\,\dot{\theta}$

$\frac{\partial L}{\partial \theta} = -md(\sin\theta\,\dot{x}\dot{\theta} + g\sin\theta = md^2(\frac{2}{5}\cos\theta\sin\theta\,\dot{\theta}^2 - \frac{g}{d}\sin\theta)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow md^2(1-\frac{2}{5}\cos^2\theta)\ddot{\theta} + \frac{2}{5}md^2\cos\theta\sin\theta\,\dot{\theta}^2 + mgd\sin\theta = 0$$

$$\Rightarrow \underline{(1-\frac{2}{5}\cos^2\theta)\ddot{\theta} + \frac{2}{5}\cos\theta\sin\theta\,\dot{\theta}^2 + \frac{g}{d}\sin\theta = 0}$$

c) Small oscillations $\theta_0 \ll 1$

approximations, keep only firstorder terms in θ and $\dot{\theta}$,

$\cos\theta \simeq 1$, $\sin\theta \simeq \theta$, $\dot{\theta}^2 \simeq 0 \Rightarrow$

$$\frac{3}{5}\ddot{\theta} + \frac{g}{d}\theta = 0, \qquad \underline{\ddot{\theta} + \frac{5}{3}\frac{g}{d}\theta = 0}$$

harmonic oscillator of the form $\ddot{\theta} + \omega^2\theta = 0$

with angular frequency $\underline{\omega = \sqrt{\frac{5g}{3d}}}$, solution $\underline{\theta = \theta_0\cos\omega t}$

Corresponding expression for x

$\dot{x} \simeq -\frac{2}{5}d\dot{\theta} \Rightarrow x = -\frac{2}{5}d\theta + x_0$

initial condition $\Rightarrow x_0 = \frac{2}{5}d\theta_0$

$$\Rightarrow \underline{x = \frac{2}{5}d\theta_0(1-\cos\omega t)}$$

Problem 2

a) Conservation of relativistic energy and momentum in S'

$$\text{I} \quad m_\tau c^2 = E_\nu' + E_K'$$

$$\text{II} \quad \vec{p}_\nu' + \vec{p}_K' = 0 \quad \Rightarrow \quad |\vec{p}_\nu'| = |\vec{p}_K'| \equiv p'$$

Energy momentum relations

$$E_K'^2 = c^2 p'^2 + m_K^2 c^4$$

$$\text{III} \quad E_\nu'^2 = c^2 p'^2 = E_K'^2 - m_K^2 c^4$$

$$\text{I} \Rightarrow \quad E_\nu'^2 = (m_\tau c^2 - E_K')^2 = E_K'^2 - 2E_K' m_\tau c^2 + m_\tau^2 c^4$$

Combined with III :

$$m_\tau^2 c^4 - 2E_K' m_\tau c^2 + m_K^2 c^4 = 0$$

$$\Rightarrow \quad E_K' = \frac{m_\tau^2 + m_K^2}{2m_\tau} c^2 = 957 \text{ MeV}$$

$$\Rightarrow \quad E_\nu' = m_\tau c^2 - E_K' = \frac{m_\tau^2 - m_K^2}{2m_\tau} c^2 = 820 \text{ MeV}$$

$$p' = \frac{E_\nu'}{c} = \frac{m_\tau^2 - m_K^2}{2m_\tau} c = 820 \text{ MeV}/c$$

In S' the situation is rotationally symmetric, no prefered direction for $\vec{p}_\nu'$

b) Velocity of τ^- i ref. frame S

$$v = 0.9c \Rightarrow \beta = 0.9, \ \gamma = 2.30$$

$$\theta_\nu' = \frac{\pi}{2} \Rightarrow \theta_K' = -\frac{\pi}{2}$$

$\Rightarrow$ in S' : $p_{\nu x}' = 0$, $p_{\nu y}' = p'$; $p_{Kx}' = 0$, $p_{Ky}' = -p'$

Lorentz-transformation from S' to S

$$E_\nu = \gamma(E'_\nu + \beta p'_{\nu x} c) = \gamma E'_\nu = \underline{1881\ \text{MeV}}$$

$$E_K = \gamma(E'_K + \beta p'_{Kx} c) = \gamma E'_K = \underline{2196\ \text{MeV}}$$

$$p_{\nu x} = \gamma(p'_{\nu x} c + \beta E'_\nu)/c = \gamma \beta E'_\nu / c = \underline{1693\ \text{MeV}/c}$$

$$p_{\nu y} = p'_{\nu y} = p' = \underline{820\ \text{MeV}/c}$$

$$p_{Kx} = \gamma(p'_{Kx} c + \beta E'_K)/c = \gamma \beta E'_K / c = \underline{1976\ \text{MeV}/c}$$

$$p_{Ky} = p'_{Ky} = \underline{-820\ \text{MeV}/c}$$

Angles

$$\theta_\nu = \text{Arctan}\left(\frac{p_{\nu y}}{p_{\nu x}}\right) = \underline{25.8°}$$

$$\theta_K = \text{Arctan}\left(\frac{p_{Ky}}{p_{Kx}}\right) = \underline{-22.5°}$$

c) All directions in S' are equally proble, which means that in this frame there is 50% probability to find the direction of the neutrino with $|\theta'_\nu| < \pi/2$.
In S this corresponds to the condition $|\theta_\nu| < 25.8°$.
The energy in S will increase with smaller angle $|\theta_\nu|$, which means larger positive component $p_{\nu x}$.
Since 50% have angles $|\theta_\nu| < 25.8°$, the same percentage will have energy $E_\nu > 1881$ MeV, which is the energy for $|\theta_\nu| = 25.8°$.

Problem 3

a) Plane wave $\vec{B}(\vec{r},t) = \frac{1}{c}\vec{n}\times\vec{E}(\vec{r},t)$; $\vec{n} = \vec{k}$

$$\Rightarrow B_y = \frac{1}{c}E_x = \frac{E_0}{c}\cos(kz-\omega t), \quad B_x = B_z = 0$$

Poynting's vector $\vec{S}_{pw} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}E_x^2\vec{k} = \frac{E_0^2}{\mu_0 c}\cos^2(kz-\omega t)\vec{k}$

b) Assume

$$\dot{x} = -\frac{eE_0}{m\omega}\sin(kz-\omega t), \quad \dot{y} = \dot{z} = 0$$

for the coordinates of the electron

$$\Rightarrow \ddot{x} = \frac{eE_0}{m}\cos(kz-\omega t), \quad \ddot{y} = \ddot{z} = 0 \Rightarrow m\vec{a} = e\vec{E}$$

Correct only if we can neglect magnetic force $e\vec{v}\times\vec{B}$ and assume non-relativistic motion.

Magnetic force: $e\vec{v}\times\vec{B} = e\dot{x}B_y\vec{k} = e\frac{\dot{x}}{c}E_x\vec{k}$, negligible if

$$|\vec{v}\times\vec{B}| << |\vec{E}| \Rightarrow \frac{|\dot{x}|}{c} << 1 \quad \text{i.e. non-relativistic speed}$$

satisfied if $\frac{eE_0}{mc\omega} << 1$

c) Time averaged emitted power from the electron

$$\overline{P} = \frac{\mu_0 e^2}{6\pi c}\overline{\vec{a}^2} \quad \text{(Larmor's formula)}$$

$$= \frac{\mu_0 e^4}{6\pi m^2 c}\overline{\vec{E}^2} = \frac{\mu_0 e^4}{12\pi m^2 c}E_0^2$$

Energy current density of the plane wave, time averaged

$$\overline{S_{pw}} = \frac{1}{\mu_0 c}\overline{\vec{E}^2} = \frac{1}{2\mu_0 c}E_0^2$$

Interaction cross section $\sigma = \frac{\overline{P}}{\overline{S_{pw}}} = \frac{\mu_0^2 e^4}{6\pi m^2}$

d) Assume electron oscillates about $x = y = z = 0$

Differential power in the radial direction

$$\frac{dP}{d\Omega} = r^2 \vec{S}_{rad} \cdot \vec{n}, \quad \vec{n} = \frac{\vec{r}}{r} \qquad \vec{S}_{rad} = \text{Poynting's vector of the radiation}$$

Assume $\vec{r}$ far from the electron

$$\vec{S}_{rad} = \frac{1}{\mu_0} \vec{E}_{rad} \times \vec{B}_{rad}$$

$$= \frac{c}{\mu_0} B_{rad}^2 \vec{n}$$

$$= \frac{c}{\mu_0} \frac{\mu_0^2 e^2}{16\pi^2 c^2} \left(\frac{\vec{a} \times \vec{n}}{r} \right)^2_{ret} \vec{n}$$

$$= \frac{\mu_0 e^2}{16\pi^2 c r^2} \left(\vec{a}^2 - (\vec{a} \cdot \vec{n}) \right)_{ret} \vec{n}$$

Introduce angle θ between $\vec{n}$ and x-axis

$$\Rightarrow \vec{a}^2 - (\vec{a} \cdot \vec{n})^2 = a^2 (1 - \cos^2\theta) = \frac{e^2 E_0^2}{m^2} \cos^2(kz - \omega t_{ret}) \sin^2\theta$$

time averaged power

$$\frac{d\bar{P}}{d\Omega} = \overline{r^2 \vec{S}_{rad} \cdot \vec{n}} = \frac{\mu_0 e^4 E_0^2}{32\pi^2 c m^2} \sin^2\theta$$

maximal for $\theta = \frac{\pi}{2}$, directions orthogonal to x-axis

minimal for $\theta = 0, \pi$ — " — along the — " —

Polarization:

$$\vec{E}_{rad}(\vec{r}, t) = \frac{\mu_0 e^2 E_0}{4\pi m} \cos(kz - \omega t_{ret}) (\vec{i} \times \vec{n}) \times \vec{n}$$

Linear polarization in direction $(\vec{n} \times \vec{i}) \times \vec{n} = \vec{i} - (\vec{n} \cdot \vec{i}) \vec{n}$