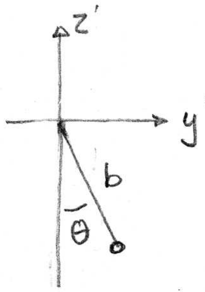
Midterm Exam FYS4110

Solutions

Problem 1

a) $x = x'\cos\omega t - y'\sin\omega t$

$y = x'\sin\omega t + y'\cos\omega t$



Coordinates of B

$x' = a\,,\quad y' = b\sin\theta\,,\quad z = -b\cos\theta$

$$\Rightarrow \quad x = a\cos\omega t - b\sin\theta\sin\omega t$$
$$y = a\sin\omega t + b\sin\theta\cos\omega t$$
$$z = -b\cos\theta$$

b) Velocity

$$\dot{x} = -a\omega\sin\omega t - b\dot{\theta}\cos\theta\sin\omega t - b\omega\sin\theta\cos\omega t$$
$$\dot{y} = a\omega\cos\omega t + b\dot{\theta}\cos\theta\cos\omega t - b\omega\sin\theta\sin\omega t$$
$$\dot{z} = b\dot{\theta}\sin\theta$$

Lagrangian

$$L = T - V = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$$
$$= \tfrac{1}{2}m\left(b^2\dot{\theta}^2 + a^2\omega^2 + b^2\omega^2\sin^2\theta + 2ab\,\omega\dot{\theta}\cos\theta\right)$$
$$+ mg\,b\cos\theta$$

$a^2\omega^2$ constant, can be absorbed in redef. of V

$$\Rightarrow \quad \underline{L = \tfrac{1}{2}mb^2\dot{\theta}^2 + mab\,\omega\dot{\theta}\cos\theta + \tfrac{1}{2}mb^2\omega^2\sin^2\theta + mgb\cos\theta}$$

c) $\frac{\partial L}{\partial \theta} = -mab\omega\dot{\theta}\sin\theta + mb^2\omega^2\sin\theta\cos\theta - mgb\sin\theta$

$\frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} + mab\omega\cos\theta$

Lagrange's equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\Rightarrow \quad mb^2\ddot{\theta} - mb^2\omega^2\sin\theta\cos\theta + mgb\sin\theta = 0$$

$$\Rightarrow \quad \underline{\ddot{\theta} + \underbrace{\frac{g}{b}\sin\theta - \omega^2\sin\theta\cos\theta}_{\equiv f(\theta)} = 0}$$

Equilibrium: $\underline{f(\theta) = 0}$

Stable equil.: $\underline{\text{— " —} \quad \& \quad \frac{df}{d\theta}(\theta) > 0}$

$f(\theta) = 0 \Rightarrow \quad \sin\theta\left(\frac{g}{b} - \omega^2\cos\theta\right) = 0$

solutions I: $\sin\theta = 0 \Rightarrow \underline{\theta = 0, \pi}$

II $\quad \cos\theta = \frac{g}{b\omega^2} \Rightarrow \underline{\theta = \pm \text{Arccos}\left(\frac{g}{b\omega^2}\right) \equiv \theta_\pm}$

II solution only if $\underline{\omega > \sqrt{\frac{g}{b}} \equiv \omega_0}$

d) Stability

$\theta = 0: \quad \frac{df}{d\theta}(0) = \omega_0^2 - \omega^2$, stable if $\underline{\omega < \omega_0}$

$\theta = \theta_\pm: \quad \cos\theta_\pm = \left(\frac{\omega_0}{\omega}\right)^2, \quad \frac{df}{d\theta}(\theta_\pm) = \omega^2\left(1 - \frac{\omega_0^4}{\omega^4}\right)$

$\frac{df}{d\theta}(\theta_\pm) > 0 \iff \underline{\omega > \omega_0}$ stable eq.

Small oscillations: frequency $\Omega^2 = \frac{df}{d\theta}(\theta_{eq})$

$\omega < \omega_0 \quad \theta = 0: \quad \underline{\Omega_0 = \sqrt{\omega_0^2 - \omega^2}}$

$\omega > \omega_0 \quad \theta = \theta_\pm: \quad \underline{\Omega_\pm = \frac{\sqrt{\omega^4 - \omega_0^4}}{\omega}}$ same frequency for θ_+ and θ_-

e) Hamiltonian

$$H = p_\theta \dot{\theta} - L = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$
$$= mb^2\dot{\theta}^2 + mb\omega\dot{\theta}\cos\theta - \tfrac{1}{2}mb^2\dot{\theta}^2 - mb\omega\dot{\theta}\cos\theta - \tfrac{1}{2}mb^2\omega^2\sin^2\theta - mgb\cos\theta$$
$$= \underline{\tfrac{1}{2}mb^2(\dot{\theta}^2 - \omega^2\sin^2\theta) - mgb\cos\theta}$$

$\neq T+V$: different from total energy, due to time dependent constraint

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0 :$$ H is a constant of motion

f) $\dot{\theta} = \frac{1}{mb^2}p_\theta - \frac{a}{b}\omega\cos\theta$

$$\Rightarrow H(p_\theta, \theta) = \tfrac{1}{2}mb^2\omega_0^2\left[\left(\frac{p_\theta}{m\omega_0 b} - \frac{a}{b}\frac{\omega}{\omega_0}\cos\theta\right)^2 - \frac{\omega^2}{\omega_0^2}\sin^2\theta - 2\cos\theta\right]$$

Introduce dimensionless variables

$$\tilde{p}_\theta = \frac{p_\theta}{m\omega_0 b}, \quad R = \frac{\omega}{\omega_0}, \quad K = \tfrac{1}{2}mb^2\omega_0^2$$

$$\Rightarrow \underline{H(\tilde{p}_\theta, \theta) = K\left[(\tilde{p}_\theta - \tfrac{a}{b}R\cos\theta)^2 - R^2\sin^2\theta - 2\cos\theta\right]}$$

In the plots: $R = 0.5, 1{,}5$; note $\frac{a}{b} = 1$, $K = 1$

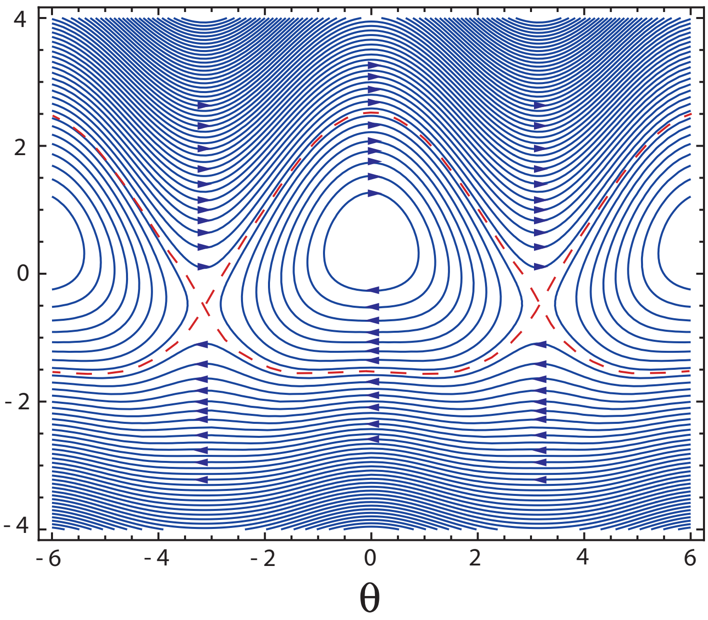
g) For $R = 0.5$: Two types of motion, oscillations about $\theta = 0$, and full rotations

For $R = 1.5$: Three types of motion, 1) Oscillations about one of the points $\theta_\pm$, 2) Oscillations about both points, 3 full rotations

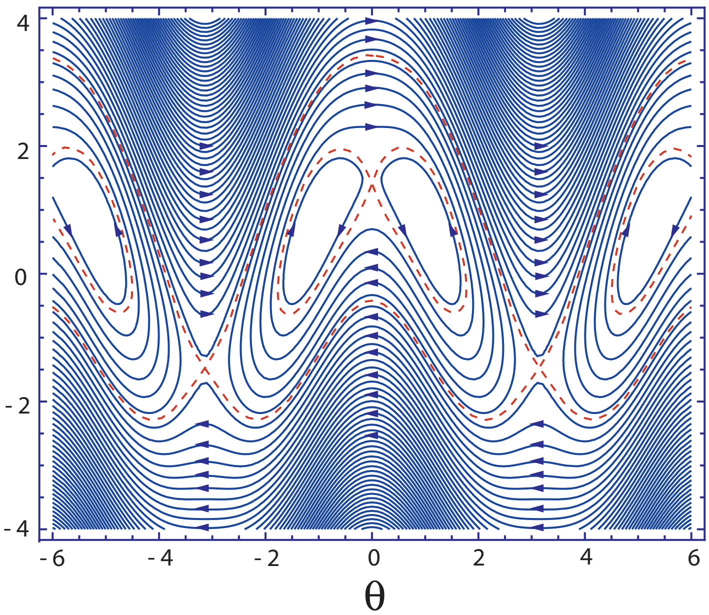
f) Phase space plot for two values of ω

Scaled momentum: $\tilde{p}_\theta = \frac{p_\theta}{m\omega_0 b}$, $a = b$

$\omega = 0.5\,\omega_0$



$\omega = 1.5\,\omega_0$



Problem 2

a) $\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2}$

$$\Rightarrow \quad v = \beta c = c\sqrt{1-\frac{1}{\gamma^2}} \simeq c\left(1-\frac{1}{2\gamma^2}\right) = \underline{0.99995\, c} \approx c$$

Period $T = \frac{2\pi R}{v} \simeq \frac{2\pi R}{c} = \frac{2 \cdot 3.14 \cdot 10}{3.0 \cdot 10^8}\, s = \underline{2.1 \cdot 10^{-7} s}$

Angular frequency $\omega = \frac{2\pi}{T} = \frac{c}{R} = \underline{3.0 \cdot 10^7 s^{-1}}$

Proper time: time measured on an imagined comoving clock

Related to coordinate by time dilatation:

$$t = \gamma \tau \Rightarrow T_\tau = \frac{1}{\gamma} T = \underline{2.1 \cdot 10^{-9}\, s}$$

b) Instant. inertial rest frame:

inertial reference frame, which at a givent instant has the same velocity as the moving body

Transformation from S to S'

$$x' = \gamma(x - vt)$$
$$y' = y + R \qquad v = \omega R$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

c) Coordinates of circular orbit in S

$$x^2 + y^2 = R^2$$

Coordinates at time $t' = 0$ in S'

$$t' = 0 \Rightarrow t = \frac{v}{c^2} x$$

$$\Rightarrow x' = \gamma\left(1 - \frac{v^2}{c^2}\right) x = \frac{1}{\gamma} x \Rightarrow \quad x = \gamma x'$$

$$y' = y + R \Rightarrow y = y' - R$$

$$\Rightarrow \quad \gamma^2 x'^2 + (y'-R)^2 = R^2 \Rightarrow \underline{\frac{x'^2}{(R/\gamma)^2} + \frac{(y'-R)^2}{R^2} = 1}$$

Ellipse with long axis R and short axis R/γ

Consistent with length contraction in x-direction

d) Particle trajectory in S

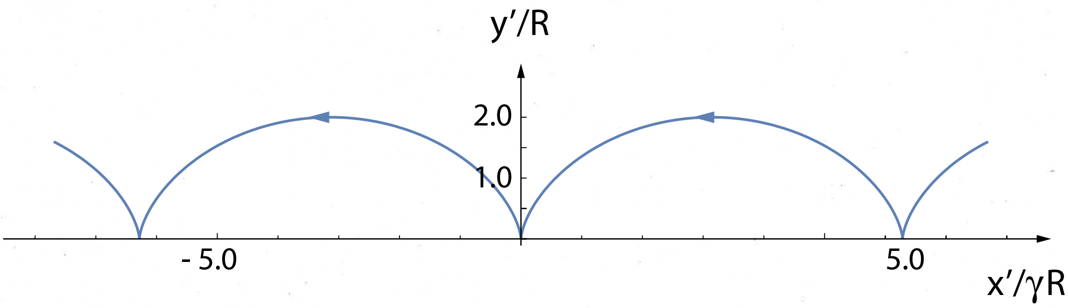
$$x = R\sin\omega t = R\sin(\gamma\omega\tau)$$
$$y = -R\cos\omega t = -R\cos(\gamma\omega t)$$
$$t = \gamma\tau$$

Trajectory in S'

$$x' = \gamma(x - vt) = \gamma(R\sin(\gamma\omega\tau) - \gamma\omega R\tau)$$
$$y' = y + R = R(1 - \cos(\gamma\omega\tau))$$
$$t' = \gamma(t - \frac{v}{c^2}x) = \gamma^2\tau - \gamma\omega R^2\frac{1}{c^2}\sin(\gamma\omega\tau)$$

Plot: Parametric plot of x' and y' with parameter $\theta = \gamma\omega\tau$

$$x'(\theta) = \gamma R(\sin\theta - \theta)$$
$$y'(\theta) = R(1 - \cos\theta)$$

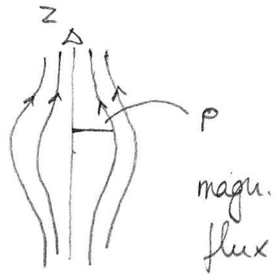


Problem 3

a) $B_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} = \underline{-\frac{1}{2}\rho B_0 \frac{df}{dz}}$

$B_\varphi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = \underline{0}$

$B_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \varphi} = \underline{B_0 f(z)}$



$\pi \rho^2 B_0 f(z) = \Phi_m$

$\Rightarrow \rho = \frac{\rho_0}{\sqrt{f(z)}} = \frac{\rho_0}{\sqrt{1+\frac{z^2}{d^2}}}$

distance of flux line from z-axis

b) $L = \frac{1}{2}mv^2 + q\vec{v}\cdot\vec{A}$

$= \underline{\frac{1}{2}m(\dot\rho^2 + \rho^2\dot\varphi^2 + \dot z^2) + \frac{1}{2}qB_0 f(z)\rho^2\dot\varphi}$

$\frac{\partial L}{\partial \dot\rho} = m\dot\rho, \quad \frac{\partial L}{\partial \rho} = m\rho\dot\varphi^2 + qB_0 f(z)\rho\dot\varphi$

$\Rightarrow \underline{m\ddot\rho - m\rho\dot\varphi^2 - qB_0 f(z)\rho\dot\varphi = 0} \quad \mathrm{I}$

$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot\varphi} = \underline{m\rho^2\dot\varphi + \frac{1}{2}qB_0 f\rho^2 = k} \text{ (const.)} \quad \mathrm{II}$

$\frac{\partial L}{\partial \dot z} = m\dot z, \quad \frac{\partial L}{\partial z} = \frac{1}{2}qB_0\frac{df}{dz}\rho^2\dot\varphi$

$\Rightarrow \underline{m\ddot z - \frac{1}{2}qB_0\frac{df}{dz}\rho^2\dot\varphi = 0} \quad \mathrm{III}$

c) $f = 1 \Rightarrow$

$\mathrm{I} \quad m\ddot\rho - m\rho\dot\varphi^2 - qB_0\rho\dot\varphi = 0$

$\mathrm{II} \quad m\rho^2\dot\varphi + \frac{1}{2}qB_0\rho^2 = k$

$\mathrm{III} \quad \ddot z = 0$

$\mathrm{III} \Rightarrow \underline{\dot z = u_0} \quad z = u_0 t$

Assume $\rho = \rho_0 \quad \dot\varphi = \omega_0$

$\mathrm{I} \Rightarrow \underline{\omega_0 = -\frac{qB_0}{m}}$

$\mathrm{II} \Rightarrow k = m\rho_0^2\omega_0 + \frac{1}{2}qB_0\rho_0^2 = \underline{-\frac{1}{2}qB_0\rho_0}$

This means $\rho = \rho_0,\ \dot\varphi = \omega_0,\ \dot z = u_0$ is a solution provided $\omega_0 = -\frac{qB_0}{m}$

d) $f = 1 + \frac{z^2}{d^2}$

Assume $\dot{\varphi} \approx -\frac{qB_z}{m}$ as for $f=1$

Introduced in the equations:

I $\ddot{\rho} \approx 0$

II $qB_z\rho^2 \simeq -2k \equiv -L_z$

III $m\ddot{z} + \frac{1}{2}(qB_0)^2 f \frac{df}{dz}\rho^2 = 0$

$$\Rightarrow m\ddot{z}\dot{z} + \frac{1}{2m}(qB_0)^2\rho^2 f\frac{df}{dz}\dot{z} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{z}^2 - \frac{1}{2m}qB_zL_z\right) = 0$$

$$\Rightarrow \frac{1}{2}m\dot{z}^2 - \frac{1}{2m}qB_zL_z = T \quad (\text{const})$$

Interpretation:

$L_z = -qB_z\rho^2 = m\rho^2\dot{\varphi} =$ z-component of particle's angular mom.

Note: $B_z\rho^2\pi = \phi_m(\rho)$ magnetic flux through orbit is conserved

$T = \frac{1}{2}m(u_0^2 + \rho_0^2\omega_0^2) =$ particle's kinetic energy

Approximation $\dot{\varphi} \simeq \frac{qB_z}{m}$ is good when d is sufficiently large, which means much larger than the other length scales in the problem: $d \gg \rho_0$, $d \gg \frac{u_0}{\omega_0}$

d) $$\frac{1}{2}m\dot{z}^2 - \frac{1}{2m}qB_zL_z = \frac{1}{2}mu_0^2 - \frac{1}{2}qB_0L_z$$

$$\Rightarrow \dot{z}^2 = u_0^2 + \frac{1}{m^2}qB_0L_z\frac{z^2}{d^2} = u_0^2 - \rho_0^2\omega_0^2\frac{z^2}{d^2}$$

Limits for z: $\dot{z} = 0$ when $z^2 = \frac{u_0^2}{\rho_0^2\omega_0^2}d^2$

$$\Rightarrow z = \pm a,$$

with $a = \frac{u_0}{\rho_0\omega_0}d$

e) Full solution

$$\dot{z}^2 = u_0^2\left(1-\frac{z^2}{a^2}\right) \Rightarrow \ddot{z}+\left(\frac{u_0}{a}\right)^2 z = 0 \quad \text{harm. osc. equation}$$

Solution with $\dot{z} = u_0$, $z = 0$ for $t = 0$

$$\underline{z(t) = a\sin\left(\frac{u_0}{a}t\right)}$$

$$\dot{\varphi} = -\frac{q}{m}B_0\left(1+\frac{z^2}{d^2}\right)$$

$$= \omega_0\left(1+\frac{a^2}{d^2}\sin^2\left(\frac{u_0}{a}t\right)\right)$$

$$= \omega_0\left(\left(1+\frac{a^2}{2d^2}\right)-\frac{a^2}{2d^2}\cos\left(2\frac{u_0}{a}t\right)\right)$$

$$\Rightarrow \varphi(t) = \underline{\omega_0\left[\left(1+\frac{a^2}{2d^2}\right)t - \frac{a^3}{4u_0 d^2}\sin\left(2\frac{u_0}{a}t\right)\right]}$$

$$\rho^2 = -\frac{L_z}{qB_0}\frac{1}{1+\frac{z^2}{d^2}} \Rightarrow \rho = \frac{\rho_0}{\sqrt{1+\frac{z^2}{d^2}}}$$

$$\underline{x(t) = \frac{\rho_0}{\sqrt{1+\frac{z(t)^2}{d^2}}}\cos(\varphi(t))}$$

$$\underline{y(t) = \frac{\rho_0}{\sqrt{1+\frac{z(t)^2}{d^2}}}\sin(\varphi(t))}$$