FYS3120 Classical mechanics and electrodynamics
Mid-term exam – Spring term 2017

Solution

April 3, 2017

Question 1 A (boring) Lagrangian

We will begin by considering a free non-relativistic particle (no potential) of mass m moving in three dimensions with no constraints.

a) Pick a sensible coordinate system for this problem and write down the Lagrangian. [1 point]

Answer: With no constraints and no potential we have three degrees of freedom and should choose the ordinary Cartesian coordinates (x, y, z) and the corresponding (component) velocities $(\dot{x}, \dot{y}, \dot{z})$. The Lagrangian $L = K - V$ will only contain a kinetic term for the particle

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{1}$$

b) Find the canonical, or conjugate, momenta for the coordinates and compare to the regular (mechanical) momentum. [1 point]

Answer: The canonical or conjugate momenta are given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}. \tag{2}$$

This is identical to the components of the mechanical momentum $\vec{\pi} = m\vec{v} = m(\dot{x}, \dot{y}, \dot{z})$.

c) Find the cyclic coordinates. [0.5 point]

Answer: Cyclic coordinates are coordinates that do not appear in the Lagrangian. In this case all the position coordinates (x, y, z).

d) What are the conserved quantities / constants of motion in this problem?[1] [0.5 point per quantity, max 3 points]

Answer: Each cyclic coordinate gives its corresponding conjugate momentum as a conserved quantity, meaning all three components of the ordinary mechanical momentum is conserved. In addition, since L has no explicit time dependence the Hamiltonian H is a constant of motion. With time-independent constraints (in our case none), the Hamiltonian is the sum of kinetic and potential energy. Thus, in our case, kinetic energy is a conserved quantity.

The Lagrangian is also unchanged under rotations in space since with $\vec{r}' = R\vec{r}$, we have $\vec{r}'^2 = \vec{r}^T R^T R\vec{r} = \vec{r}^2$. This means that the three components of angular momentum are also conserved. It is sufficient here to refer to the lectures or Sec. 2.3.4 in the notes, where the present case corresponds to the rotationally invariant potential $V(r) = 0$.

[1] The different components of a vector counts as separate quantities.

We will now repeat the above using instead a relativistic description of the same free particle.

e) Write down a Lagrangian for the corresponding relativistic case that is invariant under Lorentz transformations and demonstrate that it is indeed invariant. [1.5 points]

Answer: There are multiple possible answers here. One simple option (as discussed in the lectures) is:

$$L = \frac{1}{2} m U^{\mu} U_{\mu}, \tag{3}$$

where $U^{\mu} = (\gamma c, \gamma \vec{v})$ is the four-velocity. Since we know Lorentz vectors like U^{μ} tranform under the Lorentz transformation as

$$U^{\mu} \rightarrow U'^{\mu} = L^{\mu}{}_{\nu} U^{\nu}, \tag{4}$$

we have

$$\begin{aligned} L \rightarrow L' &= \frac{1}{2} m U'^{\mu} U'_{\mu} \\ &= \frac{1}{2} m L^{\mu}{}_{\nu} U^{\nu} L_{\mu}{}^{\rho} U_{\rho} \\ &= \frac{1}{2} m L^{\mu}{}_{\nu} L_{\mu}{}^{\rho} U^{\nu} U_{\rho} \\ &= \frac{1}{2} m \delta_{\nu}{}^{\rho} U^{\nu} U_{\rho} = \frac{1}{2} m U^{\nu} U_{\nu} = L, \end{aligned} \tag{5}$$

where we have used the requirement on the components of L given in Eq. (4.29) of the lecture notes.

f) Find the conserved quantities / constants of motion for this Lagrangian. [0.5 point per quantity, max 2 points]

Answer: The cyclic coordinates are x^{μ} since these do not appear in the Lagrangian. They lead to the conserved conjugate momenta

$$\frac{\partial L}{\partial U^{\mu}} = m U_{\mu} = (\gamma m c, \gamma m \vec{v}). \tag{6}$$

We see that the relativistic momentum $\vec{p} = \gamma m \vec{v}$ is conserved, as well as the relativistic energy $E = \gamma m c^2$. There are of course also other conserved quantities such as angular momentum. (See the next three questions.)

g) We will now look at a small Lorentz transformation where the Lorentz transformation tensor is given as

$$L^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{7}$$

Here $\delta^{\mu}{}_{\nu}$ is the Kronecker delta and $\omega^{\mu}{}_{\nu}$ is an infinitesimal parameter, meaning we can disregard higher orders of ω. Show that $\omega^{\mu}{}_{\nu}$ must be antisymmetric, *i.e.* that $\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}$. [1 point]

Answer: The Lorentz tranformation must fulfil

$$L^{\mu}{}_{\nu} L_{\mu}{}^{\rho} = \delta_{\nu}{}^{\rho}, \tag{8}$$

see for example Eq. (4.29) in the lecture notes. Inserting (7) we get

$$\begin{aligned} L^{\mu}{}_{\nu} L_{\mu}{}^{\rho} &= (\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu})(\delta_{\mu}{}^{\rho} + \omega_{\mu}{}^{\rho}) \\ &= \delta^{\mu}{}_{\nu}\delta_{\mu}{}^{\rho} + \omega^{\mu}{}_{\nu}\delta_{\mu}{}^{\rho} + \delta^{\mu}{}_{\nu}\omega_{\mu}{}^{\rho} \\ &= \delta_{\nu}{}^{\rho} + \omega^{\rho}{}_{\nu} + \omega_{\nu}{}^{\rho}, \end{aligned} \tag{9}$$

where we have ignored the term quadratic in ω. We must then have $\omega^{\rho}{}_{\nu} + \omega_{\nu}{}^{\rho} = 0$, or $\omega^{\rho}{}_{\nu} = -\omega_{\nu}{}^{\rho}$.

h) Assume that a small Lorentz transformation between two reference frames changes the path $x^{\mu}(\tau)$ of a particle according to

$$\delta x^{\mu}(\tau) = x'^{\mu}(\tau) - x^{\mu}(\tau) = \omega^{\mu}{}_{\nu} x^{\nu}(\tau). \tag{10}$$

Show that the corresponding change in the Lagrangian is

$$\delta L = \left(\frac{\partial L}{\partial x^{\mu}} x^{\nu} + \frac{\partial L}{\partial U^{\mu}} U^{\nu} \right) \omega^{\mu}{}_{\nu}. \tag{11}$$

[1 point]

Answer: The change in the four-velocity follows from (10) as

$$\delta U^{\mu}(\tau) = U'^{\mu}(\tau) - U^{\mu}(\tau) = \omega^{\mu}{}_{\nu} U^{\nu}(\tau), \tag{12}$$

since $U^{\mu} \equiv \frac{dx^{\mu}}{d\tau}$. By an expansion to first order in δx^{ν} the change in the Lagrangian is then

$$\begin{aligned} \delta L &= L' - L = L(x'^{\mu}, U'^{\mu}) - L(x^{\mu}, U^{\mu}) \\ &= L(x^{\mu}, U^{\mu}) + \frac{\partial L}{\partial x^{\mu}} \delta x^{\mu} + \frac{\partial L}{\partial U^{\mu}} \delta U^{\mu} - L(x^{\mu}, U^{\mu}) \\ &= \left(\frac{\partial L}{\partial x^{\mu}} x^{\nu} + \frac{\partial L}{\partial U^{\mu}} U^{\nu} \right) \omega^{\mu}{}_{\nu}. \end{aligned} \tag{13}$$

Here we ignore terms of higher order in $\omega^{\mu}{}_{\nu}$.

i) Use Lagrange's equation to show that you can also write Eq. (11) as

$$\delta L = \frac{1}{2}\omega_{\nu\mu}\,\frac{d}{d\tau}\left(x^\mu\frac{\partial L}{\partial U_\nu} - x^\nu\frac{\partial L}{\partial U_\mu}\right). \tag{14}$$

[1.5 points]

Answer: We again have that

$$\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial U^\mu}\delta U^\mu. \tag{15}$$

This can be rewritten

$$\begin{aligned}\delta L &= \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial U^\mu}\right)\delta x^\mu + \frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}\delta x^\mu\right)\\ &= \frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu,\end{aligned} \tag{16}$$

where we have used Lagrange's equation to eliminate a term. Since $\omega^\mu{}_\nu = -\omega_\nu{}^\mu$,

$$\omega^\mu{}_\nu = \frac{1}{2}\left(\omega^\mu{}_\nu - \omega_\nu{}^\mu\right), \tag{17}$$

we can write

$$\begin{aligned}\delta L &= \frac{1}{2}\omega^\mu{}_\nu\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right) - \frac{1}{2}\omega_\nu{}^\mu\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right)\\ &= \frac{1}{2}\omega^\mu{}_\nu\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right) - \frac{1}{2}\omega_\mu{}^\nu\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\nu}x^\mu\right)\\ &= \frac{1}{2}\omega_{\mu\nu}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\mu}x^\nu - \frac{\partial L}{\partial U_\nu}x^\mu\right)\\ &= \frac{1}{2}\omega_{\nu\mu}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\nu}x^\mu - \frac{\partial L}{\partial U_\mu}x^\nu\right),\end{aligned} \tag{18}$$

j) Identify which quantities are conserved because of the invariance under Lorentz transformations. [1.5 points]

Answer: Since the Lagrangian is Lorentz invariant we must have $\delta L = 0$, thus Eq. (14) shows that the antisymmetric tensor

$$L^{\mu\nu} \equiv x^\mu\frac{\partial L}{\partial U_\nu} - x^\nu\frac{\partial L}{\partial U_\mu} = x^\mu p^\nu - x^\nu p^\mu, \tag{19}$$

is conserved, where we have used that the generalized / canonical / conjugate momentum is given by $p^\mu = \frac{\partial L}{\partial U_\mu}$. For a free particle this is the same as the ordinary four-momentum. We will give full score for this question if you also point out that L^{ij} are the angular momenta. There is a further conserved quantity, L^{0i}, which is connected to the motion of the centre-of-mass, but elucidation of this point was not required for a full score.

Question 2 Relativistics
Two particles with mass m and a photon is sent out from a source at the same time and in the positive x-direction of the rest frame S of the source. The massive particles are moving with constant velocity v_1 and $v_2 > v_1$ in this frame. Draw a Minkowski-diagram of the motion of the source, both massive particles and the photon in S, and draw the axis of the rest frame S' of the slowest particle. Show that the difference in rapidity of the two massive particles is the same in S and S'.[2] [5 points]

Answer: The source is moving along the world-line ct since it is at rest, the photon is moving at the speed $v = c$ along the line $ct = x$, and the two massive particles lie between this, with the slowest particle closest to the ct-axis. Since the slowest massive particle is at rest in S' the ct' world-line lies in the same direction as the particle is moving in S. The x'-axis lies symmetric to the ct' axis w.r.t the light-line $ct = x$. See Fig. 1 for an illustration.

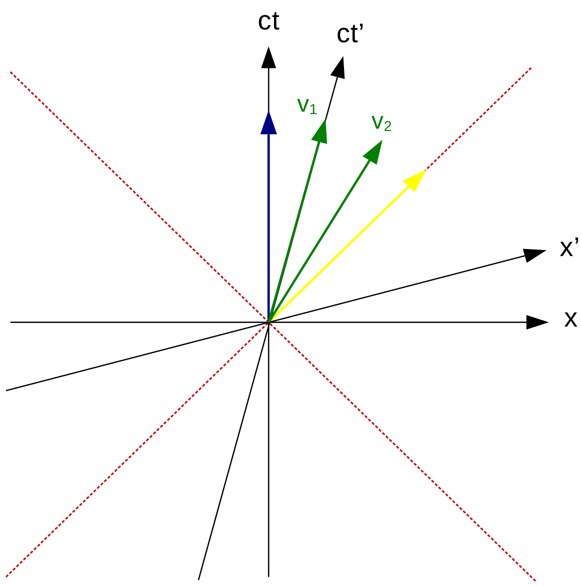

Figure 1: Minkowski diagram for emitted particles. The light-lines are shown in red, the motion of the photon in yellow, the source in blue and the massive particles in green.

In S' the velocity of the slowest massive particle is zero since this is the rest frame. The fastest massive particle has velocity v_2' given by the velocity

[2]In fact this is always true, rapidity differences are unchanged by boosts no matter which reference frames you look at. I am sure you can see this from your proof!

transformation formula

$$v_2' = \frac{v_2 - v_1}{1 - \frac{v_1 v_2}{c^2}}. \tag{20}$$

By dividing by c on both sides this can be rewritten in terms of rapidity χ with $\beta = v/c = \tanh\chi$,

$$\tanh\chi_2' = \frac{\tanh\chi_2 - \tanh\chi_1}{1 - \tanh\chi_1 \tanh\chi_2}. \tag{21}$$

From for example Rottmann we have the following relationship for tanh

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}. \tag{22}$$

This means that $\tanh\chi_2' = \tanh(\chi_2 - \chi_1)$ and thus $\chi_2' = \chi_2 - \chi_1$. Since $\chi_1' = 0$ we have achieved the desired result. This holds for all boosts by substituting v_1 for some general boost and finding the same expression for χ_1'.

Question 3 Finding the shortest way

Using calculus of variations, find the shortest path between two points on a sphere. We want the answer in terms of a function $\phi(\theta)$ using spherical coordinates (r, θ, ϕ). By the rotational symmetry of the problem you may assume that the starting point is $(r, \frac{\pi}{2}, 0)$. To simplify the answer you may also ignore the special solutions through this point with constant $\phi = 0$ or $\theta = \frac{\pi}{2}$. [5 points]

Answer: We use spherical coordinates (r, θ, ϕ) with a fixed radius r. In spherical coordinates the infinitestimal distance between two points is (see for example Rottmann):

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{23}$$

For fixed r this gives

$$ds = r\sqrt{d\theta^2 + \sin^2\theta d\phi^2} = r\sqrt{1 + \sin^2\theta\, \dot{\phi}^2}\, d\theta, \tag{24}$$

where $\dot{\phi} = d\phi/d\theta$. The total length between two points A and B is then

$$S_{AB} = \int_A^B dS = r\int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta\, \dot{\phi}^2}\, d\theta, \tag{25}$$

where we parameterise the path as a function $\phi(\theta)$. To minimize the distance as a function of the choice of path $S_{AB}[\phi(\theta)]$ through the calculus of variations we can use the function

$$L(\phi, \dot{\phi}) = \sqrt{1 + \sin^2\theta\, \dot{\phi}^2}, \tag{26}$$

as the function corresponding to the Lagrangian with θ playing the role of time t and ϕ as the generalized coordinate. This means we have to solve the following equation

$$\frac{d}{d\theta}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0. \tag{27}$$

Since L is independent on ϕ this equation can be written

$$\frac{d}{d\theta}\left(\frac{\partial}{\partial \dot{\phi}}\sqrt{1+\sin^2\theta\,\dot{\phi}^2}\right) = 0. \tag{28}$$

This means that

$$\frac{\partial}{\partial \dot{\phi}}\sqrt{1+\sin^2\theta\,\dot{\phi}^2} = \frac{\sin^2\theta\,\dot{\phi}}{\sqrt{1+\sin^2\theta\,\dot{\phi}^2}} = C \tag{29}$$

where C is some constant to be determined. This equation can be solved by squaring both sides giving

$$\frac{\sin^4\theta\,\dot{\phi}^2}{1+\sin^2\theta\,\dot{\phi}^2} = C^2, \tag{30}$$

or

$$\dot{\phi} = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}}. \tag{31}$$

Now we want to integrate this expression from the initial point where $\theta = \frac{\pi}{2}$ and $\phi = 0$ to some final point (θ, ϕ)

$$\phi(\theta) - \phi(\frac{\pi}{2}) = \int_{\frac{\pi}{2}}^{\theta} \frac{C}{\sin\eta\sqrt{\sin^2\eta - C^2}} d\eta, \tag{32}$$

where we have changed the name of the integration variable since we want to use θ to parametrise the path.

This integral is challenging. We accept solutions where this have been evaluated using formulae collections for integrals or tools such as `Mathematica` or WolframAlpha. However, we can simplify by using the substitution

$$u = \frac{C}{\sqrt{1-C^2}}\frac{1}{\tan\eta}, \quad \frac{du}{d\eta} = \frac{-C}{\sqrt{1-C^2}}\frac{1}{\sin^2\eta}. \tag{33}$$

This gives

$$\phi(\theta) = -\int_{u(\frac{\pi}{2})}^{u(\theta)} \frac{du}{\sqrt{1-u^2}}, \tag{34}$$

where we have used that

$$\sin^2\eta = \frac{1}{1+\frac{1}{\tan^2\eta}} = \frac{1}{1+\frac{1-C^2}{C^2}u^2} = \frac{C^2}{C^2+(1-C^2)u^2}. \tag{35}$$

This simpler integral has inverse sine solutions, see for example Rottmann, so

$$\phi(\theta) = \sin^{-1}(u(\frac{\pi}{2})) - \sin^{-1}(u(\theta)) = -\sin^{-1}\left(\frac{C}{\sqrt{1-C^2}}\frac{1}{\tan\theta}\right), \qquad (36)$$

using $\sin^{-1}(u(\frac{\pi}{2})) = \sin^{-1}(0) = 0$. The constant C can be determined from the initial conditions, *i.e.* the position of the endpoint. If this is (ϕ_f, θ_f) then

$$\begin{aligned}
\phi_f &= -\sin^{-1}\left(\frac{C}{\sqrt{1-C^2}}\frac{1}{\tan\theta_f}\right) \\
-\sin\phi_f &= \frac{C}{\sqrt{1-C^2}}\frac{1}{\tan\theta_f} \\
\sin^2\phi_f\tan^2\theta_f &= \frac{C^2}{1-C^2} \\
C &= \pm\sqrt{\frac{\sin^2\phi_f\tan^2\theta_f}{\sin^2\phi_f\tan^2\theta_f+1}}, \qquad (37)
\end{aligned}$$

where the sign of C has to be chosen according to the direction of the motion.