FYS3120 Classical mechanics and electrodynamics
Mid-term exam – Spring term 2018

Your **candidate number**

March 23, 2018

Question 1 Pendulum with a rotating wheel

A pendulum can rotate freely about a horizontal axis A as shown in Fig. 1. The pendulum consists of a rigid rod and attached to this is a wheel which rotates about a point B on the rod. The mass of the wheel is m and the moment of inertia about B is denoted I. We consider the mass of the pendulum rod to be negligible. The distance between the points A and B is b. The gravitational acceleration is g and the angle of the pendulum rod relative to the vertical direction is denoted ϕ. We assume that the pendulum is free to perform full rotations about the axis A.

A motor (not included in the figure) affects the rotation of the wheel by a constant angular acceleration, so that the angular velocity of the wheel measured relative to a fixed direction is $\omega = \dot{\phi} + \alpha t$, where α is the acceleration constant. For simplicity we assume that all other effects of the motor can be neglected and that friction can be disregarded.

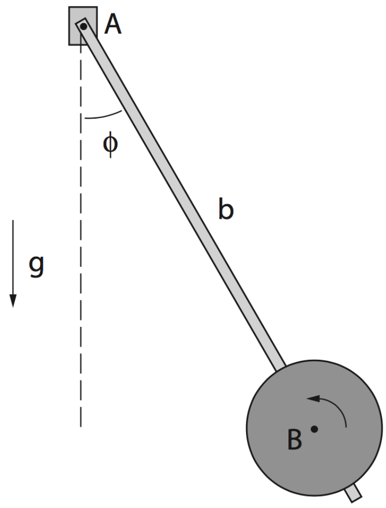

Figure 1: Pendulum with a rotating wheel.

a) Show that the Lagrangian of the system, with ϕ as a coordinate, is

$$L = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi. \tag{1}$$

[1 point]

Answer: The kinetic energy of the pendulum comes from the pendulum movement and the rotational energy of the wheel, giving

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2, \tag{2}$$

since the velocity v is related to the angular velocity of the pendulum $\dot{\phi}$ as $v = b\dot{\phi}$. The potential energy is the gravitational potential of the wheel, $V = -mgb\cos\phi$. This gives the Lagrangian

$$L = K - V = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi. \tag{3}$$

b) Find Lagrange's equation for ϕ. [2 points]

Answer: Lagrange's equation is found from

$$\begin{aligned} \frac{\partial L}{\partial \dot{\phi}} &= mb^2\dot{\phi} + I(\dot{\phi} + \alpha t) = (mb^2 + I)\dot{\phi} + I\alpha t, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= (mb^2 + I)\ddot{\phi} + I\alpha, \\ \frac{\partial L}{\partial \phi} &= -mgb\sin\phi. \end{aligned} \tag{4}$$

giving

$$(mb^2 + I)\ddot{\phi} + mgb\sin\phi + I\alpha = 0. \tag{5}$$

c) Show that you can rewrite the Lagrangian as

$$L(\phi, \dot{\phi}, t) = L'(\phi, \dot{\phi}) + \frac{d}{dt}f(\phi, t), \tag{6}$$

where

$$f(\phi, t) = I\alpha\phi t + \frac{1}{6}I\alpha^2 t^3. \tag{7}$$

[2 points]

Answer: We have

$$\frac{d}{dt}f(\phi, t) = I\alpha\dot{\phi}t + I\alpha\phi + \frac{1}{2}I\alpha^2 t^2. \tag{8}$$

giving

$$\begin{aligned} L' &= L - \frac{d}{dt}f(\phi, t) \\ &= \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi - I\alpha\dot{\phi}t - I\alpha\phi - \frac{1}{2}I\alpha^2 t^2 \\ &= \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi \\ &= \frac{1}{2}(mb^2 + I)\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi, \end{aligned} \tag{9}$$

which has no explicit dependence on time as required.

d) What are the equations of motion for the Lagrangian L'? [1 point]

Answer: The addition or subtraction of a term in the Lagrangian which is a total derivative w.r.t. time does not change the solutions of the Lagrangian equation, so the equations of motion are the same.

e) Find the canonical momentum p'_ϕ corresponding to the coordinate ϕ of the Lagrangian L' and determine the corresponding Hamiltonian $H'(\phi, p'_\phi)$. [2 points]

Answer: The canonical momentum is

$$p'_\phi = \frac{\partial L'}{d\dot{\phi}} = (mb^2 + I)\dot{\phi}. \tag{10}$$

The Hamiltonian is

$$\begin{aligned} H'(\phi, p'_\phi) &= p'_\phi \dot{\phi} - L' \\ &= (mb^2 + I)\dot{\phi}\dot{\phi} - \frac{1}{2}(mb^2 + I)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi \\ &= \frac{1}{2}(mb^2 + I)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi \\ &= \frac{p'^2_\phi}{2(mb^2 + I)} - mgb(\cos\phi - \frac{I}{mgb}\alpha\phi). \end{aligned} \tag{11}$$

f) Explain why H' is a constant of motion. [1 point]

Answer: There is no explicit time-dependence in H', thus $dH'/dt = \partial H'/\partial t = 0$ and H' is a constant of motion.

g) Make a two dimensional contour plot of the phase space potential function $H'(\phi, p'_\phi)$, for different values of

$$\lambda = \frac{I}{mgb}\alpha, \tag{12}$$

for example for $\lambda = 0, 0.5, 1.0$. Make sensible choices for the other parameters. Give a qualitative description of the different types of motion that can be read out of the diagrams and comment on how the situation changes with increasing λ. [3 points]

Answer: We show the phase space for $\lambda = 0, 0.5, 1.0$ in Fig. 2. We observe the following motion:

$\lambda = 0$ There are three types of motion: oscillations about a stable equilibrium corresponding to circles in phase space, full rotation of the pendulum in the positive ϕ direction, and full rotation of the pendulum in the negative direction. There is one stable equilibrium at $\phi = 0$ and one unstable at $\phi = \pi$.

$0 < \lambda < 1$ There are two types of motion: oscillations about the stable equilibrium and full rotation of the pendulum in the negative direction (rotation of the pendulum in the positive direction will turn into rotation in the negative direction).

$\lambda > 1$ One type of motion: rotation of the pendulum in the negative direction with no oscillations possible.

Question 2 Two-body decays

In particle physics we are often interested in two-body decays, where one heavy particle decays into two other lighter particles, for example the top quark t can decay into a W-boson and a bottom quark b, a process that we symbolise as $t \to bW$. Here we would like to study generic two-body decays of the form $B \to aA$.

a) Below we will use the concept of *invariant mass*. For two particles a and b the invariant mass m_{ab} is given by

$$m_{ab}^2 c^2 = (p_a + p_b)^2 = (p_a + p_b)^\mu (p_a + p_b)_\mu, \tag{13}$$

where p_a and p_b are the four-momenta of the particles. Explain why the invariant mass does not change between reference frames. [1 point]

Answer: The sum of two Lorentz vectors is a Lorentz vector because the Lorentz transformation acts linearly. The invariant mass is thus a contraction of a Lorentz vector with itself and is therefore invariant under changes of reference frame.

b) For the decay $B \to aA$, find the magnitude of the relativistic momentum of particle a in the rest frame of A expressed in terms of the masses of the particles (and c). [3 points]

Answer: Conservation of four-momenta gives $p_B^\mu = p_a^\mu + p_A^\mu$. Contracting this four-vector with itself yields

$$m_B^2 c^2 = m_a^2 c^2 + m_A^2 c^2 + 2 p_a^\mu p_{A\mu} = m_a^2 c^2 + m_A^2 c^2 + 2(E_a E_A / c^2 - \vec{p}_a \vec{p}_A), \tag{14}$$

which in the rest frame of A gives

$$m_B^2 c^2 = m_a^2 c^2 + m_A^2 c^2 + 2\sqrt{|\vec{p}_a|^2 + m_a^2 c^2}\, m_A c. \tag{15}$$

This can be solved for p_a giving

$$\begin{aligned}\sqrt{|\vec{p}_a|^2 + m_a^2c^2} &= \frac{m_B^2c^2 - m_A^2c^2 - m_a^2c^2}{2m_Ac} \\ |\vec{p}_a|^2 &= \frac{(m_B^2c^2 - m_A^2c^2 - m_a^2c^2)^2 - 4m_A^2m_a^2c^4}{4m_A^2c^2},\end{aligned} \quad (16)$$

or

$$|\vec{p}_a| = \frac{\sqrt{(m_B^2 - m_A^2 - m_a^2)^2 - 4m_A^2m_a^2}}{2m_A}c. \quad (17)$$

c) In the two sequential two-body decays $C \to bB$ and $B \to aA$, using four-vectors and invariants, find the square of the invariant mass of a and b, m_{ab}^2, expressed by the particle masses and the angle θ between a and b in the rest frame of B. To simplify the calculation you may assume $m_a = m_b = 0$. [5 points]

Answer: The invariant mass is given as

$$\begin{aligned}m_{ab}^2c^2 &= (p_a + p_b)^2 = p_a^2 + p_b^2 + 2p_ap_b \\ &= m_a^2c^2 + m_b^2c^2 + 2(E_aE_b/c^2 - \vec{p}_a\vec{p}_b) \\ &= 2(|\vec{p}_a||\vec{p}_b| - \cos\theta|\vec{p}_a||\vec{p}_b|) = 2(1-\cos\theta)|\vec{p}_a||\vec{p}_b|,\end{aligned} \quad (18)$$

where in the last line we have used that a and b are massless, and where θ is the angle between the particles. We can here take over the result in **b)** giving with an appropriate change in names

$$|\vec{p}_b| = \frac{\sqrt{(m_C^2 - m_B^2 - m_b^2)^2 - 4m_B^2m_b^2}}{2m_B}c = \frac{m_C^2 - m_B^2}{2m_B}c, \quad (19)$$

in the rest frame of B. For the decay of B we have again $p_B^\mu = p_a^\mu + p_A^\mu$, but now, as we want to work in the rest frame of B, we square p_A since we are disinterested in its momentum:

$$\begin{aligned}p_A^2 &= p_A^\mu p_{A\mu} = m_A^2c^2 = (p_B - p_a)^2 = p_B^2 + p_a^2 - 2p_Bp_a \\ &= m_B^2c^2 - 2(E_BE_a/c^2 - \vec{p}_B\vec{p}_a) \\ &= m_B^2c^2 - 2m_B|\vec{p}_a|c,\end{aligned} \quad (20)$$

where we have used that b is massless and that we are in the rest frame of B. This is easily solved for $|\vec{p}_a|$ giving the strikingly symmetric

$$|\vec{p}_a| = \frac{m_B^2 - m_A^2}{2m_B}c. \quad (21)$$

Input in the invariant mass we arrive at

$$m_{ab}^2 = \frac{(m_C^2 - m_B^2)(m_B^2 - m_A^2)}{m_B^2}\frac{1}{2}(1-\cos\theta). \quad (22)$$

d) For a chain of four sequential two-body decays, $E \to dD$, $D \to cC$, $C \to bB$ and $B \to aA$, write a code to numerically find the distribution of invariant masses of all possible combinations of pairs of particles a, b, c and d. Plot these distributions in the same figure. For masses you should choose $m_E = 600$ GeV/c^2, $m_D = 500$ GeV/c^2, $m_C = 200$ GeV/c^2, $m_B = 150$ GeV/c^2, $m_A = 100$ GeV/c^2, $m_d = m_c = 0$ and $m_b = m_a = 1.8$ GeV/c^2. You can assume that all the decays are *isotropic*, *i.e.* that the direction of the decay products in the rest frame of the decaying particle is uniformly distributed on a sphere.[1] It may be wise to use recursive function calls for the decays. No points will be given for submitted code. [3 points]

Answer: We show the invariant mass distributions in Fig. 3. The code used to generate this plot can be found in the `python` script `decay_chain.py`.

The philosophy behind the code is the following: we can pick any frame to do the calculation as we are after invariants. We may as well choose the reference frame where the initial particle, E, is at rest. We can then take over the result in **b)** to find the momenta (and energies) of the decay products and choose their direction from a uniform (on a sphere) distribution. The kinematics of the decay products of D can likewise be found in the rest frame of D, and then boosted back to the rest frame of E. This way we can continue down the chain until we know the momenta and energies of all the particles in the decay. To find the invariant mass distributions all we need to do is to histogram many such decays.

[1] Note that there is a very nasty but educational trap here.

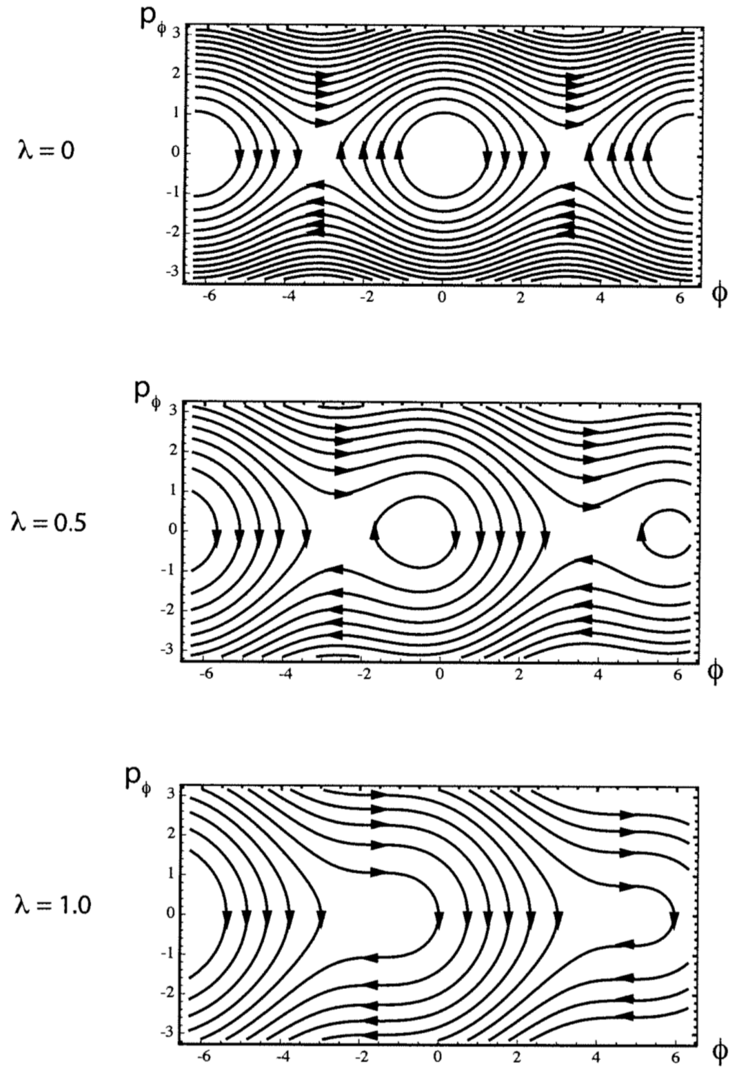

Figure 2: Phase space for $\lambda = 0$ (top), $\lambda = 0.5$ (middle), and $\lambda = 1$ (bottom).

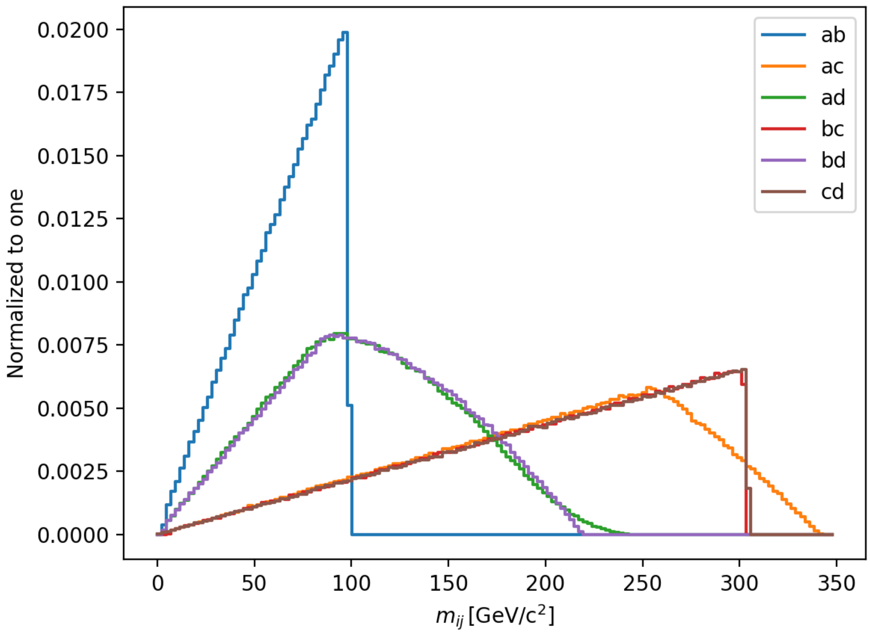

Figure 3: Invariant mass distributions for all pairs of particles.