Proof that the force from an external magnetic field $\vec{B}$ on a static current density $\vec{j}(\vec{r})$ is given by

$$\vec{F}_m = (\vec{m}\cdot\vec{\nabla}')\vec{B}(\vec{r}')\Big|_{\vec{r}'=0} + \dots$$

We have shown in lecture that

$$\vec{F}_m = \int \vec{j}(\vec{r}) \times (\vec{r}\cdot\vec{\nabla}')\vec{B}(\vec{r}')\, d^3\vec{r} + \dots$$

Now we use that

$$(\vec{r}\cdot\vec{\nabla}')\vec{B}(\vec{r}') = \vec{\nabla}'(\vec{r}\cdot\vec{B}(\vec{r}')) - \vec{r}\times\left[\vec{\nabla}'\times\vec{B}(\vec{r}')\right]$$

this can be seen from Rottmann p. 64

$$\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{b}\cdot\vec{\nabla})\vec{a} + (\vec{a}\cdot\vec{\nabla})\vec{b} + \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a})$$

with $\vec{a}=\vec{r}$ and $\vec{b}=\vec{B}(\vec{r}')$ and using that $\vec{\nabla}'\times\vec{r}=0$ and $(\vec{B}(\vec{r}')\cdot\vec{\nabla}')\vec{r}=0$ since the differentiation is w.r.t. $\vec{r}'$.
Further $\vec{\nabla}'\times\vec{B}(\vec{r}')=0$ since $\vec{B}$ is external and has no source near $\vec{j}(\vec{r})$ (Ampère's law)

Thus $$\int \vec{j}(\vec{r})\times(\vec{r}\cdot\vec{\nabla}')\vec{B}(\vec{r}')\, d^3\vec{r} = \int \vec{j}(\vec{r})\times\vec{\nabla}'(\vec{r}\cdot\vec{B}(\vec{r}'))\, d^3\vec{r}$$

$$= -\int \vec{\nabla}'\times(\vec{r}\cdot\vec{B}(\vec{r}'))\vec{j}(\vec{r})\, d^3\vec{r} = -\vec{\nabla}'\times\int(\vec{r}\cdot\vec{B}(\vec{r}'))\vec{j}(\vec{r})\, d^3\vec{r} \qquad (*)$$

We can now write (tripple cross product rule, p. 62 Rottmann)

$$-\int(\vec{r}\cdot\vec{B}(\vec{r}'))\vec{j}(\vec{r})\, d^3\vec{r} = \int \vec{B}(\vec{r}')\times(\vec{r}\times\vec{j}(\vec{r}))\, d^3\vec{r} - \int \vec{r}\,(\vec{B}(\vec{r}')\cdot\vec{j}(\vec{r}))\, d^3\vec{r} \qquad (**)$$

Furthermore $$\int \vec{r}\,(\vec{B}(\vec{r}')\cdot\vec{j}(\vec{r}))\, d^3\vec{r} = -\int(\vec{r}\cdot\vec{B}(\vec{r}'))\vec{j}(\vec{r})\, d^3\vec{r} \qquad (***)$$

This follows from the same argument used to find the magnetic dipole contribution to the vector potential, namely that $\int \vec{r}\,'(\vec{j}(\vec{r}\,')\cdot\vec{r})\, d^3\vec{r}\,' = -\int \vec{j}(\vec{r}\,')(\vec{r}\,'\cdot\vec{r})\, d^3\vec{r}\,'$

From (*), (**) and (***)

$$\vec{F}_m = \frac{1}{2}\vec{\nabla}'\times\int\vec{B}(\vec{r}')\times(\vec{r}\times\vec{j}(\vec{r}))\, d^3\vec{r} = \frac{1}{2}\vec{\nabla}'\times\vec{B}(\vec{r}')\times\int\vec{r}\times\vec{j}(\vec{r})\, d^3\vec{r}$$

$$= \vec{\nabla}' \times \vec{B}(\vec{r}\,') \times \vec{m} = (\vec{m} \cdot \vec{\nabla}')\vec{B}(\vec{r}\,') - (\vec{B}(\vec{r}\,') \cdot \vec{\nabla}')\vec{m} + \vec{B}(\vec{r}\,')(\vec{\nabla}' \cdot \vec{m}) - \vec{m}(\vec{\nabla}' \cdot \vec{B}(\vec{r}\,'))$$

$$= (\vec{m} \cdot \vec{\nabla}')\vec{B}(\vec{r}\,') - \vec{m}(\vec{\nabla}' \cdot \vec{B}(\vec{r}\,'))$$

Here we have used (Rottmann p. 64) that

$$\vec{\nabla} \times \vec{a} \times \vec{b} = (\vec{b} \cdot \vec{\nabla})\vec{a} - (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{a}(\vec{\nabla} \cdot \vec{b}) - \vec{b}(\vec{\nabla} \cdot \vec{a})$$

with $\vec{a} = \vec{B}(\vec{r}\,')$ and $\vec{b} = \vec{m}$ and the fact that $\vec{m}$ is a constant.

Since $\vec{\nabla}' \cdot \vec{B}(\vec{r}\,') = 0$ (no monopoles)

$$\vec{F}_m = (\vec{m} \cdot \vec{\nabla}')\vec{B}(\vec{r}\,')$$

q.e.d.