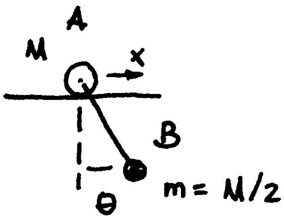
Exam FYS 3120/4120 Spring semester 2006
Solutions

Problem 1



$I = \frac{1}{2} M R^2$

$\dot{x} = \omega R$

(rolling cylinder)

a) Kinetic energy

$$T_A = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \omega^2 = \frac{3}{4} M \dot{x}^2$$

$$T_B = \frac{1}{2} m\left(\underbrace{(\dot{x} + L\cos\theta\,\dot{\theta})^2}_{\dot{x}_B^2} + \underbrace{L^2 \sin^2\theta\,\dot{\theta}^2}_{\dot{y}_B^2}\right) = \frac{1}{4} M(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\cos\theta\,\dot{\theta}\,\dot{x})$$

Potential energy

$$V_A = 0\,,\quad V_B = -mgL\cos\theta = -\frac{M}{2} gL\cos\theta$$

Lagrangian

$$L = T - V = \underline{M\left(\dot{x}^2 + \frac{1}{4}L^2\dot{\theta}^2 + \frac{1}{2}L\cos\theta\,\dot{\theta}\,\dot{x} + \frac{1}{2}gL\cos\theta\right)}$$

b) $\frac{\partial L}{\partial x} = 0 \Rightarrow x$ cyclic $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$

$$K \equiv \frac{\partial L}{\partial \dot{x}} = \underline{M(2\dot{x} + \frac{1}{2}L\cos\theta\,\dot{\theta})} \quad \text{is constant of motion}$$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0$$

$$\Rightarrow H = T + V = \underline{M(\dot{x}^2 + \frac{1}{4}L^2\dot{\theta}^2 + \frac{1}{2}L\cos\theta\,\dot{\theta}\,\dot{x} - \frac{1}{2}gL\cos\theta)} \equiv E \text{ (energy)}$$

is constant of motion

c) $$\frac{\partial L}{\partial \dot{\theta}} = M\left(\frac{1}{2}L^2\dot{\theta} + \frac{1}{2}L\cos\theta\,\dot{x}\right) \qquad \frac{\partial L}{\partial \theta} = M\left(-\frac{1}{2}L\sin\theta\,\dot{\theta}\,\dot{x} + \frac{1}{2}gL\sin\theta\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad \underline{\ddot{\theta} + \frac{1}{L}\cos\theta\,\ddot{x} + \frac{g}{L}\sin\theta = 0}$$

~~The equation discussed under b)~~

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\left(2\ddot{x} + \tfrac{1}{2}L\cos\theta\,\ddot{\theta} - \tfrac{1}{2}L\sin\theta\,\dot{\theta}^2\right) = 0$$

$$\Rightarrow \quad \ddot{x} = -\tfrac{1}{4}L\cos\theta\,\ddot{\theta} + \tfrac{1}{4}L\sin\theta\,\dot{\theta}^2$$

Inserted in the θ-equation:

$$\left(1 - \tfrac{1}{4}\cos^2\theta\right)\ddot{\theta} + \tfrac{1}{4}\sin\theta\cos\theta\,\dot{\theta}^2 + \tfrac{g}{\ell}\sin\theta = 0$$

Small angle approximation $\theta \ll 1 \Rightarrow \cos\theta \simeq 1$, $\sin\theta \simeq \theta$

Approximation to first order in $\theta, \dot{\theta}, \ddot{\theta}$:

$$\left(1 - \tfrac{1}{4}\right)\ddot{\theta} + \tfrac{g}{\ell}\theta = 0 \quad \Rightarrow \quad \underline{\ddot{\theta} + \tfrac{4}{3}\tfrac{g}{\ell}\theta = 0}$$

Harmonic osc. equation with frequency $\omega = \sqrt{\frac{4}{3}\frac{g}{\ell}}$

General solution $\theta(t) = A\cos(\omega t - \varphi)$

Initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0 \Rightarrow \underline{\theta(t) = \theta_0\cos\omega t}$

Solution for the x-coordinate

$$\dot{x}(0) = \dot{\theta}(0) = 0 \Rightarrow K = 0$$

$$\Rightarrow \quad \dot{x} = -\tfrac{1}{4}L\cos\theta\,\dot{\theta} \simeq -\tfrac{1}{4}L\dot{\theta}$$

$$\Rightarrow x = -\tfrac{1}{4}L\theta + x_0 \quad \text{const}$$

$t = 0$ $\quad x_0 = \frac{1}{4}L\theta_0 \Rightarrow x(t) = +\frac{1}{4}L(\theta_0 - \theta(t))$

$$= \tfrac{1}{4}L\theta_0(1 - \cos\omega t) = \underline{\tfrac{1}{2}L\theta_0\sin^2\tfrac{\omega t}{2}}$$

Problem 2

a) Equation of motion

$$m_e(\gamma\vec{a} + \dot{\gamma}\vec{v}) = e\vec{E}$$

Instantaneous inertial frame moves along $\vec{E}$:
no transformation of $\vec{E}$

In this frame: $\vec{v} = 0$, $\gamma = 1$, $\dot{\gamma} = 0$, $\vec{a} \equiv \vec{a}_0$

$$\Rightarrow \underline{m_e\vec{a}_0 = e\vec{E}}$$

b) $$\frac{d}{dt}(\gamma\vec{v}) = \frac{e\vec{E}}{m_e} \quad \Rightarrow \quad \gamma\vec{v} = \frac{e\vec{E}}{m_e}t + \underbrace{\text{const}}_{=0}$$

$$\Rightarrow \gamma^2\beta^2 = \left(\frac{eE}{m_e c}\right)^2 t^2 \qquad \beta = \frac{v}{c}$$

$$\gamma^2 = \frac{1}{1-\beta^2} \quad \Rightarrow \quad \gamma^2\beta^2 = \gamma^2 - 1$$

$$\gamma^2 = \gamma^2\beta^2 + 1 = \left(\frac{eE}{m_e c}\right)^2 t^2 + 1$$

$$\Rightarrow \gamma = \sqrt{1+\kappa^2 t^2} \qquad \underline{\kappa = \frac{eE}{m_e c} = \frac{a_0}{c}}$$

c)

$$\gamma = \cosh\kappa\tau$$

$$\Rightarrow \kappa^2 t^2 = \cosh^2\kappa\tau - 1 = \sinh^2\kappa t \quad \Rightarrow \quad t = \frac{1}{\kappa}\sinh\kappa\tau$$

$$\frac{dt}{d\tau} = \cosh\kappa\tau \; \underline{= \gamma}$$

Compare definition of proper time

$$d\tau^2 = dt^2 - \frac{1}{c^2}dx^2 = \left(1 - \frac{v^2}{c^2}\right)dt^2 = \frac{1}{\gamma^2}dt^2$$

or $$\underline{\frac{dt}{d\tau} = \gamma}$$

Problem 3

a) Electric dipole radiation (formula coll.)

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi c r}\ddot{\vec{p}}(t_{ret})\times\vec{n}, \quad t_{ret} = t - \frac{r}{c}, \quad \vec{n} = \frac{\vec{r}}{r}$$

$$\ddot{\vec{p}} = -\omega^2\vec{p} = -\omega^2 l q\,(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j})$$

$$\vec{n} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k} \qquad \text{polar coord.}$$

$$\Rightarrow \vec{B}(\vec{r},t) = -\frac{\mu_0\omega^2 l q}{4\pi c r}(\cos\theta\sin\omega t_{ret}\,\vec{i} - \cos\theta\cos\omega t_{ret}\,\vec{j} - \sin\theta\sin(\omega t_{ret}-\varphi)\,\vec{k})$$

$$\underline{B_0 = -\frac{\mu_0\omega^2 l q}{4\pi c r}}$$

For radiation fields (formula coll.) $\underline{\vec{E}(\vec{r},t) = c\vec{B}(\vec{r},t)\times\vec{n}}$

b) Radiation in the x-direction; $\vec{n} = \vec{i}$ $(\theta = \frac{\pi}{2}, \varphi = 0)$:

$\vec{B}(\vec{r},t) = -B_0\sin\omega t_{ret}\,\vec{k}$, oscillates in the z-direction

$\Rightarrow$ <u>linear polarization</u>

Radiation in the z-direction; $\vec{n} = \vec{k}$ $(\theta = 0)$

$\vec{B}(\vec{r},t) = B_0(\sin\omega t_{ret}\,\vec{i} - \cos\omega t_{ret}\,\vec{j})$, rotates in the x,y-plane

$\Rightarrow$ <u>circular polarization</u>

c) Energy density

$$u = \frac{1}{2}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) = \frac{1}{\mu_0}\vec{B}^2 = \frac{1}{\mu_0}B_0^2(\cos^2\theta + \sin^2\theta\sin^2(\omega t_{ret}-\varphi))$$

time average $\overline{\sin^2\omega t} = \frac{1}{2}$ $\Rightarrow$ $\bar{u} = \frac{1}{2\mu_0}B_0^2(1+\cos^2\theta)$

for fixed r maximal for $\cos\theta = \pm 1$; <u>along the z-axis</u>

This is the direction of maximal radiation, since radiated energy is proportional to u:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{c}{\mu_0}B^2\vec{n} = \underline{c u \vec{n}}$$