Eksamen FYS 3120 / 4120 v. 2007

Løsninger

Oppgave 1

a) Lagrangefunksjon

$$L = \tfrac{1}{2}(2m)(\dot r^2 + r^2\dot\varphi^2) + \tfrac{1}{2}m\dot r^2 + mg(d-r)$$

$$= \underline{\tfrac{3}{2}m\dot r^2 + mr^2\dot\varphi^2 + mg(d-r)}$$

b) φ - variabel : $\frac{\partial L}{\partial \varphi} = 0$, $\frac{\partial L}{\partial \dot\varphi} = 2mr^2\dot\varphi$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = 0 \Rightarrow \underline{\frac{d}{dt}(2mr^2\dot\varphi) = 0}$$

r - variabel : $\frac{\partial L}{\partial r} = 2mr\dot\varphi^2 - mg$, $\frac{\partial L}{\partial \dot r} = 3m\dot r$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \underline{3m\ddot r - 2mr\dot\varphi^2 + mg = 0}$$

φ syklisk : $\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \ell = 2mr^2\dot\varphi$ = konstant

$$\dot\varphi = \frac{\ell}{2mr^2} \Rightarrow \underline{3m\ddot r - \frac{\ell^2}{2mr^3} + mg = 0}$$

Tolkning av ℓ : $\ell = 2m(\vec r \times \dot{\vec r})_z$, banespinn til klossen

c) Sirkelbevegelse : $r = r_0$ (konstant) , $\ddot r = 0$

$$\Rightarrow \frac{\ell^2}{2mr_0^3} = mg \quad , \quad \underline{r_0 = \sqrt[3]{\frac{\ell^2}{2m^2 g}}}$$

Små svingninger om r_0 , $r = r_0 + \rho$, $\ddot r = \ddot\rho$

Rekkeutvikling til 1. orden : $\frac{1}{r^3} = \frac{1}{r_0^3} - 3\frac{\rho}{r_0^4} + \cdots$

$$3m\ddot\rho + 3\frac{\ell^2}{2mr_0^4}\rho = 0 \Rightarrow \ddot\rho + \omega_\rho^2\rho = 0$$

svingeligning med sirkelfrekvens $\underline{\omega_\rho = \frac{\ell}{\sqrt{2}\, m r_0^2}}$

(Sml. vinkel~~frekvens~~ hastighet $\omega = \dot\varphi = \frac{\ell}{2mr_0^2}$)

Oppgave 2

a) $\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2}, \; \beta = \sqrt{1-\frac{1}{\gamma^2}} \simeq 1 - \frac{1}{2\gamma^2} \quad (\gamma \gg 1)$

Elektronhastighet $v = \beta c = \underline{0.99995c} \simeq c$

Vinkelhastighet $\omega = \frac{v}{R} \simeq \frac{c}{R} = \underline{3.0 \cdot 10^7\, s^{-1}}$

Akselerasjon $a = \frac{v^2}{R} = \omega^2 R = \underline{9.0 \cdot 10^{15}\, m/s^2}$

b) Egentid $d\tau = \frac{1}{\gamma} dt \Rightarrow t = \gamma\tau$ (γ konstant)

Koordinater (setter $t=0$ ved $y = x^2 = 0$)

$x^0 = ct = \underline{\gamma c\tau}$, $x^1 = R\cos\omega t = \underline{R\cos\gamma\omega\tau}$

$x^2 = R\sin\omega t = \underline{R\sin\gamma\omega\tau}$, $\underline{x^3 = 0}$

4-hastighet $U^\mu = \frac{dx^\mu}{d\tau} \Rightarrow$

$U^0 = \underline{\gamma c}$, $U^1 = \underline{-\gamma\omega R\sin\gamma\omega\tau}$, $U^2 = \underline{\gamma\omega R\cos\gamma\omega\tau}$, $U^3 = \underline{0}$

4-akselerasjon $A^\mu = \frac{dU^\mu}{d\tau} \Rightarrow$

$A^0 = \underline{0}$, $A^1 = \underline{-\gamma^2\omega^2 R\cos\gamma\omega\tau}$, $A^2 = \underline{-\gamma^2\omega^2 R\sin\gamma\omega\tau}$, $A^3 = \underline{0}$

Akselerasjon i momentant hvilesystem

$(A^\mu) = (0, \vec a_0) \Rightarrow \vec a_0^{\,2} = A^\mu A_\mu = (\gamma^2\omega^2 R)^2$, $a_0 = \gamma^2\omega^2 R = \underline{10^4 a}$

c) Transformasjon til momentant hvilesystem:

$\vec v$ = elektronhastighet

Lab. system: $\vec E = 0$, $\vec B = \vec B_\perp$

Mom. hvilesystem: $\vec B' = \vec B'_\perp = \underline{\gamma\vec B}$, $\vec E' = \vec E'_\perp = \underline{\gamma\vec v\times\vec B}$

Bev. lign. i labsystem: $\gamma m\vec a = e\vec v\times\vec B$

Mom. hvilesystem $\vec a_0 = \gamma^2\vec a \Rightarrow m\vec a_0 = \gamma^2 m\vec a = e\gamma\vec v\times\vec B$

$\Rightarrow \underline{m\vec a_0 = e\vec E'}$

stemmer med bev. lign når $\vec v = 0$ (i mom. hvilesyst.)

Oppgave 3

a) Elektrisk dipolmoment $\vec{p} = \int \vec{r} \rho \, dV = 0$ siden $\rho = 0$

Magnetisk dipolmoment $\vec{m} = \frac{1}{2} \int \vec{r} \times \vec{j} \, dV$

Integrert over tverrsnitt: $\vec{j} \, dV \to j A \, dl \, \vec{e}_\varphi$

A = tverrsnittareal, $dl = a \, d\varphi$, $\vec{e}_\varphi$ = enhetsvektor langs sløyfen

Strøm $I = jA \Rightarrow$

$$\vec{m} = \frac{I}{2} \int_0^{2\pi a} a \, \vec{e}_r \times \vec{e}_\varphi \, dl \qquad \vec{e}_r \times \vec{e}_\varphi = \vec{e}_z$$

$$= \frac{I}{2} a \, 2\pi a \, \vec{e}_z$$

$$= \underline{\pi a^2 I_0 \cos\omega t \, \vec{e}_z} \quad \Rightarrow \quad \underline{m_0 = \pi a^2 I_0}$$

b) Strålingsfelt på x-aksen, $r = x$, $\vec{n} = \vec{e}_x$ (antar $x > 0$)

$$\ddot{\vec{m}} = -\omega^2 m_0 \cos\omega t \, \vec{e}_z$$

$$\ddot{\vec{m}} \times \vec{n} = -\omega^2 m_0 \cos\omega t \, \vec{e}_z \times \vec{e}_x = -\omega^2 m_0 \cos\omega t \, \vec{e}_y$$

$$\vec{E}(\vec{r}, t) = -\frac{\mu_0}{4\pi c r} \ddot{\vec{m}}_{ret} \times \vec{n}$$

$$= -\frac{\mu_0}{4\pi c x} (-\omega^2 m_0) \cos(\omega(t - \tfrac{x}{c})) \, \vec{e}_y$$

$$= \underline{\frac{\mu_0 \omega^2 a^2 I_0}{4 c x} \cos(kx - \omega t) \, \vec{e}_y} \qquad k = \frac{\omega}{c}$$

$$\vec{B}(\vec{r}, t) = -\frac{1}{c} \vec{E}(\vec{r}, t) \times \vec{e}_x = \underline{\frac{\mu_0 \omega^2 a^2 I_0}{4 c^2 x} \cos(kx - \omega t) \, \vec{e}_z}$$

Lineært polarisert langs y-aksen ($\vec{E}$-felt)

c) Poyntings vektor

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} \vec{E}^2 \vec{e}_x \qquad \text{strålingsfelt på x-aksen}$$

Effekt: $dP = \vec{S} \cdot d\vec{A} = S r^2 d\Omega$

$$\frac{dP}{d\Omega} = S r^2 = \frac{x^2}{\mu_0 c} E^2 = \underline{\frac{\mu_0 \omega^4 a^4 I_0^2}{16 c^3} \cos^2(kx - \omega t)}$$

Strålingsfelt på z-aksen

$$\vec{E} = \vec{B} = 0 \Rightarrow \underline{\left(\frac{dP}{d\Omega}\right)_{z\text{-akse}} = 0}$$