Exam FYS 3120, spring 2008

Solutions

Problem 1

Constraint $z = \frac{1}{2}\lambda(x^2+y^2) = \frac{1}{2}\lambda r^2 \Rightarrow \dot{z} = \lambda r \dot{r}$

$x = r\cos\theta, \; y = r\sin\theta \Rightarrow \dot{x}^2+\dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$

a) Lagrangian

$$L = T - V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$$

$$= \underline{\frac{1}{2}m\left(\dot{r}^2(1+\lambda^2 r^2) + r^2\dot{\theta}^2 - g\lambda r^2\right)}$$

Lagrange's equations

$$\theta: \quad \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \underline{\frac{d}{dt}(mr^2\dot{\theta}) = 0} \qquad \mathrm{I}$$

$$\Rightarrow \underline{\dot{\theta} = \frac{\alpha}{r^2}} \qquad \alpha \text{ constant}$$

$$r: \quad \frac{\partial L}{\partial r} = m(\lambda^2 r\dot{r}^2 + r\dot{\theta}^2 - g\lambda r), \quad \frac{\partial L}{\partial \dot{r}} = m(1+\lambda^2 r^2)\dot{r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \underline{(1+\lambda^2r^2)\ddot{r} + \lambda^2 r\dot{r}^2 - r\dot{\theta}^2 + g\lambda r = 0} \qquad \mathrm{II}$$

b) θ cyclic $\Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ constant $\Rightarrow \dot{\theta} = \frac{\alpha}{r^2}$ $\quad\alpha$ constant

Inserted in II: $\underline{(1+\lambda^2r^2)\ddot{r} + \lambda^2 r\dot{r}^2 - \frac{\alpha^2}{r^3} + g\lambda r = 0}$ radial eq.

Circular motion: $r = r_0, \; \dot{r} = \ddot{r} = 0$

$$\Rightarrow -\frac{\alpha^2}{r_0^3} + g\lambda r_0 = 0 \Rightarrow \frac{\alpha^2}{r_0^4} = g\lambda \Rightarrow \underline{\dot{\theta} = \frac{\alpha}{r_0^2} = \sqrt{g\lambda}}$$

c) Small deviations: $r = r_0 + \rho, \; \dot{r} = \dot{\rho}, \; \ddot{r} = \ddot{\rho}$

Eq. to first order in ρ: $\frac{1}{r^3} = \frac{1}{r_0^3}\frac{1}{(1+\frac{\rho}{r_0})^3} \simeq \frac{1}{r_0^3}(1 - 3\frac{\rho}{r_0})$

$$\Rightarrow (1+\lambda^2 r_0^2)\ddot{\rho} + 3\frac{\alpha^2}{r_0^4}\rho + g\lambda\rho = 0$$

$\frac{\alpha^2}{r_0^4} = g\lambda \Rightarrow \quad \underline{\ddot{\rho} + \Omega^2\rho = 0} \quad$ with $\underline{\Omega = 2\sqrt{\frac{g\lambda}{1+\lambda^2 r_0^2}}}$

harm. osc. eq. $\qquad$ circular frequency

Solution $\rho(t) = \rho_0 \cos(\Omega(t-t_0))$

$$\Rightarrow \quad z(t) = \frac{1}{2}\lambda r_0^2 + \lambda r_0 \rho_0 \cos(\Omega(t-t_0))$$

$$\dot\theta(t) = \frac{\alpha}{r^2} \approx \frac{\alpha}{r_0^2} - 2\sqrt{g\ell}\,\frac{\rho_0}{r_0}\cos(\Omega(t-t_0))$$

Qualitative descr.: small oscillations in the z-coordinate of the orbit, combined with small oscillations in the angular velocity of the particle around the constant value α/r_0^2.

Problem 2

a) Conservation of relativistic energy and momentum in S

$$\text{I} \quad m_\pi c^2 = E_\mu + E_\nu$$

$$\text{II} \quad 0 = \vec p_\mu + \vec p_\nu \quad \Rightarrow \quad |\vec p_\mu| = |\vec p_\nu|$$

π – pion
μ – muon
ν – neutrino

Energy-momentum relations

$$E_\mu^2 = \vec p_\mu^{\,2} c^2 + m_\mu^2 c^4$$

$$E_\nu^2 = \vec p_\nu^{\,2} c^2 = \vec p_\mu^{\,2} c^2 = E_\mu^2 - m_\mu^2 c^4$$

In eq. I:

$$(m_\pi c^2 - E_\mu)^2 = E_\mu^2 - m_\mu^2 c^4$$

$$\Rightarrow \quad E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} c^2 = 215\, m_e c^2 = \underline{109.6\ \text{MeV}}$$

Momentum

$$p_\mu^2 = \frac{E_\mu^2}{c^2} - m_\mu^2 c^2 = \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi} c\right)^2$$

$$\Rightarrow \quad |\vec p_\mu| = |\vec p_\nu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c = 58.0\, m_e c = \underline{29.6\ \text{MeV}/c}$$

$$E_\nu = |\vec p_\mu|\, c = \underline{29.6\ \text{MeV}}$$

b) Transformation from S to $\bar{S}$

$$\bar{p}^0 = \gamma(p^0 + \beta p^1) \qquad \bar{p}^2 = p^2$$
$$\bar{p}^1 = \gamma(p^1 + \beta p^0) \qquad \bar{p}^3 = p^3$$

Valid for both muon and neutrino with $p^0 = \frac{E}{c}$, $p^1 = p_x$ etc.

Energies in $\bar{S}$ with $\theta = \frac{\pi}{2} \Rightarrow p_x = 0$, $p_y = p$

$$\bar{E}_\mu = \gamma E_\mu = \frac{5}{3} E_\mu = \underline{182.7 \text{ MeV}} \qquad \beta = \frac{4}{5} \Rightarrow \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{5}{3}$$
$$\bar{E}_\nu = \gamma E_\nu = \frac{5}{3} E_\nu = \underline{49.3 \text{ MeV}}$$

c) In S: Probability $= 0.5$ for directions with $\theta < \frac{\pi}{2} \equiv \theta_0$

(same probability for $\theta > \frac{\pi}{2}$)

In $\bar{S}$: Probability $= 0.5$ for $\bar{\theta} < \bar{\theta}_0$ where $\bar{\theta}_0$ is the direction in $\bar{S}$ corresponding to θ_0 in S

$$\tan\bar{\theta}_0 = \frac{\bar{p}_{\nu y}}{\bar{p}_{\nu x}} = \frac{p_{\nu y}}{\gamma\beta p_\nu^0} = \frac{p_\nu}{\gamma\beta p_\nu} = \frac{1}{\gamma\beta} = \frac{3}{4}$$
$$\Rightarrow \bar{\theta}_0 = \arctan\frac{3}{4} = \underline{36.9^\circ}$$

Problem 3

a) Circular motion in magnetic field

$$m\vec{a} = e\vec{v} \times \vec{B}_0 \Rightarrow \dot{\vec{v}} = \vec{\omega} \times \vec{v} \quad \text{with } \vec{\omega} = -\frac{e}{m}\vec{B}_0, \quad \vec{B}_0 = B_0 \vec{e}_z$$

angular frequency: $\vec{\omega} = \omega \vec{k}_z$ with $\omega = -\frac{eB_0}{m}$

$$a^2 = \omega^2 v^2 = \omega^4 r^2 \quad (\vec{B}_0 \cdot \vec{v} = 0)$$

Larmor:

radiated power $P = \frac{\mu_0 e^2}{6\pi c} a^2 = \underline{\frac{\mu_0 e^2}{6\pi c} \omega^4 r^2}$

b) Energy conservation

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = -P$$

$$\Rightarrow \quad \tfrac{1}{2}m\omega^2 \frac{d}{dt} r^2 = -\frac{\mu_0 e^2}{6\pi c}\omega^4 r^2$$

Of the form

$$\frac{d}{dt} r^2 = -2\lambda r^2 \quad \text{with } \underline{\lambda = \frac{\mu_0 e^2}{6\pi c}\frac{\omega^2}{m}}$$

Solution :

$$\frac{dr^2}{r^2} = -2\lambda\, dt \Rightarrow \ln r^2 = -2\lambda t + \underbrace{\text{const}}_{\equiv \ln r_0^2}$$

$$\Rightarrow \quad r^2 = r_0^2 e^{-2\lambda t}, \quad \underline{r = r_0 e^{-\lambda t}}$$

c) Electric dipole moment

$$\underline{\vec{p} = e\vec{r}} \quad \Rightarrow \quad \ddot{\vec{p}} = e\vec{a}$$

Electric dipole radiation : On the z axis, distance $= |z|$

$\Rightarrow$ retarded time $t_r = t - \frac{|z|}{c} = t \mp \frac{z}{c}$ with $z = \pm |z|$

Magnetic field (Cartesian unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$):

$$\vec{B}(z,t) = -\frac{\mu_0}{4\pi c|z|}\vec{e}_z \times \ddot{\vec{p}}(t_r)$$

Note : r = electron orbit radius

Electron orbit $\vec{r}(t) = r(\cos\omega t\, \vec{e}_x + \sin\omega t\, \vec{e}_y)$

$$\Rightarrow \quad \vec{a}(\vec{r}) = -\omega^2 r \left(\text{— " —} \right)$$

$$\Rightarrow \quad \underline{\vec{B}(z,t) = \frac{\mu_0 e}{4\pi c}\frac{r}{|z|}\omega^2\left(-\sin(\omega t \mp kz)\vec{e}_x + \cos(\omega t \mp kz)\vec{e}_y\right)} \qquad k = \frac{\omega}{c}$$

Electric field

$$\vec{E}(z,t) = c\vec{B}\times\vec{e}_z = \underline{\frac{\mu_0 e}{4\pi}\frac{r}{|z|}\omega^2\left(\cos(\omega t \mp kz)\vec{e}_x + \sin(\omega t \mp kz)\vec{e}_y\right)}$$

propagating waves in direction $\pm \vec{e}_z$

Polarization : <u>circular, righthanded (for positive ω)</u>