Exam FYS 3120, spring 2009
Solutions

Problem 1

a) Coordinates of pendulum bob

$$x = R\cos\omega t + l\sin\theta \;; \qquad y = R\sin\omega t - l\cos\theta$$

Velocities

$$\dot{x} = -\omega R\sin\omega t + \dot{\theta} l\cos\theta \;; \qquad \dot{y} = \omega R\cos\omega t + \dot{\theta} l\sin\theta$$

Kinetic energy

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}m(\dot{\theta}^2 l^2 + 2\dot{\theta}\omega R l\sin(\theta - \omega t) + \omega^2 R^2)$$

Potential energy

$$V = mgy = -mgl\cos\theta + mgR\cancel{\cos\omega}\sin\omega t$$

Lagrangian

$$\underline{L = T - V = m\left[\tfrac{1}{2}\dot{\theta}^2 l^2 + \dot{\theta}\omega R l\sin(\theta - \omega t) + gl\cos\theta - gR\sin\omega t + \tfrac{1}{2}\omega^2 R^2\right]}$$

b) Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m(l^2\dot{\theta} + \omega R l\sin(\theta - \omega t))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m(l^2\ddot{\theta} + \omega R l(\dot{\theta} - \omega)\cos(\theta - \omega t))$$

$$\frac{\partial L}{\partial \theta} = m(\dot{\theta}\omega R l\cos(\theta - \omega t) - gl\sin\theta)$$

$$\Rightarrow \quad l^2\ddot{\theta} + gl\sin\theta - \omega^2 R l\cos(\theta - \omega t) = 0$$

$$\underline{\ddot{\theta} + \frac{g}{l}\sin\theta - \omega^2\frac{R}{l}\cos(\theta - \omega t) = 0}$$

c) Small $|\theta|$ and ω:

$$|\theta| \ll 1 \Rightarrow \sin\theta \simeq \theta, \quad \cos\theta \simeq 1$$

$$\cos(\theta - \omega t) = \cos\theta \cos\omega t + \sin\theta \sin\omega t$$

$$\simeq \cos\omega t + \theta \sin\omega t$$

$$\omega^2 \cos(\theta - \omega t) \simeq \omega^2 \cos\omega t$$

(neglect term $\sim \omega^2\theta$ as higher order in small quantities)

Equation of motion (approximation for small $|\theta|, \omega$)

$$\underline{\ddot{\theta} + \frac{g}{\ell}\theta = \omega^2 \frac{R}{\ell} \cos\omega t}$$

Driven harmonic oscillator, subject to periodic force $\sim \cos\omega t$

Particular solution $\theta(t) = \theta_0 \cos\omega t$

$$\Rightarrow \quad \left(-\omega^2 + \frac{g}{\ell}\right)\theta_0 = \omega^2 \frac{R}{\ell}$$

def.: $\omega_0^2 = \frac{g}{\ell} \quad \Rightarrow \quad \underline{\theta_0 = \frac{R}{\ell} \frac{\omega^2}{\omega_0^2 - \omega^2}}$

Resonance for $\omega = \omega_0$

Condition of small ω:

$$\theta_0 \ll 1 \Rightarrow \omega^2 \ll \frac{\ell}{R}(\omega_0^2 - \omega^2)$$

$$\Rightarrow \omega^2 \ll \frac{\omega_0^2}{1 + \frac{R}{\ell}} = \underline{\frac{g}{\ell + R}}$$

Problem 2

a) Equation of motion

$$\frac{d\vec{p}}{dt} = q\vec{E}$$

Solution : $\underline{\vec{p}(t) = \vec{p}_0 + q\vec{E}\,t}$

Energy $\mathcal{E}^2 = c^2p^2 + m^2c^4$

$$= c^2p_0^2 + m^2c^4 + (qEct)^2 \qquad \vec{p}_0 \cdot \vec{E} = 0$$

$$= \mathcal{E}_0^2 + (qEt)^2$$

$$\Rightarrow \quad \underline{\mathcal{E}(t) = \sqrt{\mathcal{E}_0^2 + (qEct)^2}}$$

Relativistic gamma factor

$$\mathcal{E} = \gamma mc^2 \Rightarrow \quad \underline{\gamma(t) = \frac{\mathcal{E}_0}{mc^2}\sqrt{1 + \left(\frac{qE}{\mathcal{E}_0}ct\right)^2}}$$

b) Velocity

$$\vec{v} = \frac{\vec{p}}{\gamma m} \Rightarrow v_x = \frac{p_0}{\gamma m} = \underline{\frac{p_0c^2}{\mathcal{E}_0}\frac{1}{\sqrt{1 + \left(\frac{qE}{\mathcal{E}_0}ct\right)^2}}}$$

$$v_y = \frac{qEt}{\gamma m} = \underline{\frac{c^2}{\mathcal{E}_0}\frac{qEt}{\sqrt{1 + \left(\frac{qE}{\mathcal{E}_0}ct\right)^2}}}$$

No force in the x-direction $\Rightarrow$ p_x = constant

Velocity v_x decreases, since the relativistic mass γm increases when v_y increases.

c) $$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = v_x\gamma = \frac{p_0}{m}$$

$$d\tau = \frac{m}{p_0}dx \Rightarrow \underline{\Delta\tau = \frac{m}{p_0}L} \qquad \alpha = \frac{m}{p_0}$$

d) $$\frac{dt}{d\tau} = \gamma = \frac{\varepsilon_0}{mc^2}\sqrt{1+\left(\frac{qE}{\varepsilon_0}ct\right)^2}$$

$$\Rightarrow \int_0^{\Delta t} \frac{dt}{\sqrt{1+\left(\frac{qE}{\varepsilon_0}ct\right)^2}} = \frac{\varepsilon_0}{mc^2}\int_0^{\Delta\tau} d\tau = \frac{\varepsilon_0}{mc^2}\Delta\tau = \frac{\varepsilon_0 L}{p_0 c^2}$$

Change variable $\xi = \frac{qE}{\varepsilon_0}ct$

$$\int_0^{\frac{qE}{\varepsilon_0}c\Delta t} \frac{d\xi}{\sqrt{1+\xi^2}} = \frac{qEL}{p_0 c^{\cancel{2}}}$$

$$= \operatorname{arc\,sinh}\left(\frac{qE}{\varepsilon_0}c\Delta t\right)$$

$$\Rightarrow \underline{\Delta t = \frac{\varepsilon_0}{qEc}\sinh\left(\frac{qEL}{p_0 c^{\cancel{2}}}\right)}$$

Problem 3

a) Charge conservation $\Rightarrow$ continuity equation

$$\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

ρ - volume density of charge, $\vec{j}$ current density

Assume: Current runs only in the z-direction

$$\nabla\cdot\vec{j} = \frac{dj}{dz} \; ; \quad j = j_z$$

Integrate over cross-section of antenna

$$I = \int j\,dA \qquad \text{current}$$

$$\lambda = \int \rho\,dA \qquad \text{linear charge density}$$

$$\Rightarrow \quad \frac{\partial\lambda}{\partial t} + \frac{\partial I}{\partial z} = \int\left(\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial z}\right)dA \; \underline{= 0}$$

Current $I(z,t) = I_o \cos\frac{\pi z}{a} \cos\omega t$

$$\Rightarrow \quad \frac{\partial\lambda}{\partial t} = -\frac{\partial I}{\partial z} = \frac{\pi}{a} I_o \sin\frac{\pi z}{a} \cos\omega t$$

$$\Rightarrow \quad \lambda(z,t) = \underline{\frac{\pi}{\omega a} I_o \sin\frac{\pi z}{a} \sin\omega t} + f(z)$$

Initial condition

$$\lambda(z,0) = f(z) = 0$$

b) Electric dipole moment

$$\vec{p} = \int \vec{r}\rho\,d^3r = \int_{-a/2}^{a/2} z\,\lambda(z,t)\,dz\,\vec{k}$$

$$= \frac{\pi}{\omega a} I_o \sin\omega t\,\vec{k} \int_{-a/2}^{a/2} z \sin\frac{\pi z}{a}\,dz$$

$$\int_{-a/2}^{a/2} z\sin\frac{\pi z}{a}\,dz = \underbrace{\left[-\frac{a}{\pi} z\cos\frac{\pi z}{a}\right]_{-a/2}^{a/2}}_{=0} + \frac{a}{\pi}\int_{-a/2}^{a/2}\cos\frac{\pi z}{a}\,dz$$

$$= \left(\frac{a}{\pi}\right)^2 \left[\sin\frac{\pi z}{a}\right]_{-a/2}^{a/2} = 2\left(\frac{a}{\pi}\right)^2$$

$$\Rightarrow \quad \vec{p}(t) = \underline{2\frac{a I_o}{\omega\pi}\sin\omega t\,\vec{k}} \qquad \underline{p_o = \frac{2aI_o}{\omega\pi}}$$

c) Radiation fields (electric dipole radiation)

$$\vec{B}(\vec{r},t) = -\frac{\mu_0}{4\pi c r}\,\vec{n}\times\ddot{\vec{p}}(t-\tfrac{r}{c}) \qquad \vec{n} = \frac{\vec{r}}{r}$$

$$\vec{E}(\vec{r},t) = c\,\vec{B}(\vec{r},t)\times\vec{n}$$

Here : $\ddot{\vec{p}}(t-\frac{r}{c}) = -\omega^2 p_0\,\vec{k}\,\sin(\omega(t-\frac{r}{c}))$

On the x-axis : $\vec{r} = r\vec{i}$

$$\vec{B}(\vec{r},t) = -\frac{\mu_0}{4\pi c r}(-\omega^2 p_0)\sin(\omega(t-\tfrac{r}{c}))\,\vec{i}\times\vec{k}$$

$$= -\frac{\mu_0\omega^2 p_0}{4\pi c r}\sin(\omega(t-\tfrac{r}{c}))\,\vec{j}$$

$$= -\frac{\cancel{2}\mu_0 a I_0 \omega}{2\pi^2 c r}\sin(\omega(t-\tfrac{r}{c}))\,\vec{j}$$

$$E(r\vec{i},t) = \frac{\mu_0 a I_0 \omega}{2\pi^2 c r}\sin(\omega(t-\tfrac{r}{c}))\,\vec{k}$$

The radiation in the direction of the x-axis is linearly polarized (plane polarized) in the direction of the z-axis.