FYS 3120, Exam 2013

Solutions

Problem 1

a) Lagrangian

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) - mgr\sin\omega t$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}\,, \quad \frac{\partial L}{\partial r} = mr\omega^2 - mg\sin\omega t$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \ddot{r} - r\omega^2 + g\sin\omega t = 0$$

b) Particular solution

$$r = k\sin\omega t \Rightarrow \ddot{r} = -k\omega^2\sin\omega t$$

$$\Rightarrow \ddot{r} - r\omega^2 = -2k\omega^2\sin\omega t$$

Eq. of motion satisfied, provided $2k\omega^2 = g \Rightarrow k = \frac{1}{2}\frac{g}{\omega^2}$

Cartesian coordinates

$$x = r\cos\omega t = k\cos\omega t\sin\omega t = \frac{1}{2}k\sin 2\omega t$$

$$y = r\sin\omega t = k\sin^2\omega t = \frac{1}{2}k(1-\cos 2\omega t)$$

$$\Rightarrow x^2 + (y - \tfrac{1}{2}k)^2 = (\tfrac{1}{2}k)^2$$

Circle with radius $\frac{1}{2}k = \frac{1}{4}\frac{g}{\omega^2}$

Center at $x = 0$, $y = \frac{1}{2}k = \frac{1}{4}\frac{g}{\omega^2}$

Velocity

$$\left.\begin{aligned} v_r &= \dot{r} = \omega k \sin\omega t \\ v_\varphi &= r\dot{\varphi} = \omega k \cos\omega t \end{aligned}\right\} \Rightarrow v = \sqrt{v_r^2 + v_\varphi^2} = \omega k = \underline{\frac{1}{2}\frac{g}{\omega}}$$

constant speed

c) Homogeneous equation

$$\ddot{r} - r\omega^2 = 0$$

has solutions of the form $e^{\lambda t}$ with $\lambda^2 = \omega^2 \Rightarrow \lambda = \pm\omega$

General solution $r = Ae^{\omega t} + Be^{-\omega t}$

General solution of the full equation

$$\underline{r = Ae^{\omega t} + Be^{-\omega t} + k\sin\omega t}$$

A, B determined by initial conditions, $k = \frac{1}{2}\frac{g}{\omega^2}$

Types of motion

$A = 0$, r will approach, exponentially fast, the bounded, particular solution $r = k\sin\omega t$

$A \neq 0$, r will exponentially fast approach $\pm\infty$

Since an arbitrarily small A will lead to this divergence, the particular solution is (dynamically) unstable

Problem 2

a) Transformation between S and S'

$$x' = \gamma(x - \beta ct) \qquad \beta = \frac{v}{c} \quad \gamma = (1-\beta^2)^{-1/2}$$

$$t' = \gamma(t - \frac{\beta}{c}x)$$

$$\Rightarrow x'^2 - c^2t'^2 = \gamma^2(x-\beta ct)^2 - \gamma^2c^2(t-\frac{\beta}{c}x)^2$$

$$= \gamma^2(1-\beta^2)(x^2 - c^2t^2) = x^2 - c^2t^2$$

$$x_A^2 - c^2t^2 = \frac{c^4}{a^2} \quad \Rightarrow \quad x_A'^2 - c^2t'^2 = \frac{c^4}{a^2}$$

$$\Rightarrow x_A' = \sqrt{c^2t'^2 + \frac{c^4}{a^2}} = \underline{c\sqrt{t'^2 + \frac{c^2}{a^2}}}$$ similarly for B

b) Velocities in S

$$v_A = \frac{dx_A}{dt} = \frac{ct}{\sqrt{t^2 + \frac{c^2}{a^2}}}, \quad v_B = \frac{dx_B}{dt} = \frac{ct}{\sqrt{t^2 + \frac{c^2}{b^2}}}$$

$$v_A = 0 \Rightarrow t = 0 \Rightarrow v_B = 0$$

Both v_A and v_B vanish at $t=0$, means S is instant. inert rest frame for A and B at $t=0$

Distance between A and B at $t=0$,

$$d = x_B - x_A = \frac{c^2}{b} - \frac{c^2}{a} = \underline{c^2\frac{a-b}{ab}}$$

Similarly for distance in S', measured at $t'=0$

$$d' = x_B' - x_A' = \underline{c^2\frac{a-b}{ab}} = d \qquad \text{same distance}$$

On the trajectory of A $t' = 0$ corresponds to a different point than $t = 0$: $t' = 0 \Rightarrow x_A' = \frac{c^2}{a}$

Expressed in terms of t and x_A:

$$\left.\begin{aligned} \gamma\left(t - \tfrac{\beta}{c} x_A\right) &= 0 \\ \gamma\left(x_A - \beta c t\right) &= \frac{c^2}{a} \end{aligned}\right\} \Rightarrow \gamma t\left(1-\beta^2\right) = \frac{c}{a} \Rightarrow \underline{t = \beta\gamma \frac{c}{a}}$$

By varying β (velocity of S') t can take any value, means that $t' = 0$ corresponds to any point on the trajectory of A.

This implies the conclusions, since S' can be the instantaneous inertial rest frame of A for any point on its trajectory.

c) Proper acceleration is the acceleration measured in the instantaneous inertial rest frame.

Measured in S' at $t' = 0$:

$$a_A'(t) = \frac{dv_A'}{dt'} = \frac{c}{\sqrt{t'^2 + \frac{c^2}{a^2}}} - \frac{c\, t'^2}{\left(\sqrt{t'^2 + \frac{c^2}{a^2}}\right)^3} \Rightarrow a_A'(0) = \underline{a}$$

Implies the proper acceleration is $\underline{a}$ for any point on the trajectory of A.

Similarly b is the proper acceleration for any point on the trajectory of B.

d) Emission of signal at A : $t_A = 0$, $x_A = \frac{c^2}{a}$ (I)

Reception of signal at B at t_B , $x_B^2 = c^2 t_B^2 + \frac{c^4}{b^2}$ (II)

Lightlike separation of the two events :

$$(x_B - x_A)^2 = c^2 (t_B - t_A)^2$$

$$\Rightarrow (x_B - \frac{c^2}{a})^2 = c^2 t_B^2 = x_B^2 - \frac{c^4}{b^2} \quad \text{(by use of I and II)}$$

Simplifies to

$$x_B = \frac{1}{2} a c^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

$$t_B = \frac{1}{2} a c \left(\frac{1}{b^2} - \frac{1}{a^2} \right)$$

Velocity of B :

$$v_B = \frac{c t_B}{\sqrt{t_B^2 + \frac{c^2}{b^2}}} = c \, \frac{c t_B}{x_B} = \underline{c \, \frac{a^2 - b^2}{a^2 + b^2}}$$

Consider light signal as photon emitted from A, with $E = h\nu_0$ as energy measured in rest frame of A, received with energy $E' = h\nu'$ measured in rest frame of B

Corresponding fourmomenta

$$p_0 = \frac{E}{c} , \ p_x = p_0 ; \quad p_0' = \frac{E'}{c} , \ p_x' = p_0' \quad (p_y = p_z = 0)$$

Lorents transformation

$$p_0' = \gamma (p_0 - \beta p_x) = \gamma (1 - \beta) p_0 = \sqrt{\frac{1-\beta}{1+\beta}} \, p_0$$

with $\beta = v_B / c$ and $p_0' = \frac{h}{c} \nu'$ $\Rightarrow$

$$\nu' = \sqrt{\frac{1-\beta}{1+\beta}} \, \nu_0 = \underline{\frac{b}{a} \nu_0}$$

Problem 3

a) Magnetic dipole moment

$$\vec{m} = \frac{1}{2}\int(\vec{r}\times\vec{j})\,dV \qquad \vec{j} = j\,\vec{e}_\varphi \quad j = \frac{I}{\Delta A}\ \text{cross-section area}$$

$$= \frac{1}{2}a^2 I \int_0^{2\pi}(\vec{e}_r\times\vec{e}_\varphi)\,d\varphi \qquad \vec{e}_r\times\vec{e}_\varphi = \vec{e}_z$$

$$= \pi a^2 I\,\vec{e}_z$$

$$= \pi a^2 I_0 \cos\omega t\,\vec{e}_z \quad \Rightarrow \quad \underline{m_0 = \pi a^2 I_0}$$

For large r ($r \gg a$, $r \gg \lambda = 2\pi c/\omega$) the radiation fields, which go like $1/r$ dominate. For $\lambda \gg a$ ($\omega a \ll 1$) the multipole expansion converges rapidly. Here: the current loop is charge neutral ⇒ electric dipole and quadrupole moments vanish. Therefore magnetic dipole contribution dominates.

b) $\ddot{\vec{m}}_{ret} = -\pi a^2\omega^2 I_0\cos(\omega(t-\frac{r}{c}))$; $\omega(t-\frac{r}{c}) = \omega t - kr$, $k = \frac{\omega}{c}$

$$\Rightarrow \vec{E}(\vec{r},t) = \frac{\mu_0 a^2}{4cr} I_0\,\omega^2\cos(\omega t - kr)\,\vec{e}_z\times\vec{n} \ ; \quad \vec{e}_z\times\vec{n} = \sin\theta\,\vec{e}_\varphi$$

$$\vec{B}(\vec{r},t) = -\frac{\mu_0 a^2}{4c^2 r} I_0\,\omega^2\cos(\omega t - kr)\sin\theta\,\vec{e}_\theta \ ; \quad \vec{e}_\theta = -\vec{n}\times\vec{e}_\varphi$$

Functional dependence $\sim f(\omega t - kr)$, for large r plane wave which propagates with speed $c = \frac{\omega}{k}$ in the radial direction.

$\vec{E}$ (and $\vec{B}$) oscillates with orientation in fixed direction

⇒ linear polarization. Polarization plane spanned by $\vec{n}$ and $\vec{e}_\varphi$

c) Radiated power

$$\frac{dP}{d\Omega} = r^2 S \qquad S = \frac{1}{\mu_0}(\vec{E}\times\vec{B})\cdot\vec{n} = \frac{1}{\mu_0 c}\vec{E}^2$$

$$\underline{\frac{dP}{d\Omega} = \frac{\mu_0 a^4}{16c^3} I_0^2 \omega^4 \sin^2\theta \cos^2\omega t}$$

Averaged over time: $\cos^2\omega t \rightarrow \frac{1}{2}$

Integrated over angles

$$\bar{P} = \int \frac{d\bar{P}}{d\Omega} d\Omega = 2\pi \frac{\mu_0 a^4}{32c^3} I_0^2 \omega^4 \int_0^\pi \sin^3\theta \, d\theta$$

$$= \underline{\frac{\mu_0 \pi a^4}{12c^3} I_0^2 \omega^4}$$

d) Current in the second loop caused by induced emf., given by Faraday's law

$$\mathcal{E} \equiv \oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\Phi_m \,, \quad \Phi_m = \int_S \vec{B}\cdot d\vec{A} = \int_S \vec{B}\cdot\vec{u}\, dA$$

Maximum amplitude in the current oscillations when $|\vec{B}\cdot\vec{u}|$ is maximal. In the x,y plane:

$\sin\theta = 1 \quad \vec{e}_\theta = -\vec{e}_z \Rightarrow |\vec{B}\cdot\vec{u}|$ maximal when $\underline{\vec{u} = \pm\vec{e}_z}$

This means that the second loop should, like the first loop lie in the x,y-plane.