Exam FYS3120, 2014

Solutions

Problem 1

Lagrangian,

a) $L = \frac{1}{2}m\vec{v}^2 - V(\vec{r}) + q\vec{v}\cdot\vec{A}$

$= \frac{1}{2}m\dot{\vec{r}}^2 - \frac{1}{2}m\omega_0^2 r^2 - \frac{1}{2}q\vec{v}\cdot(\vec{r}\times\vec{B})$

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{1}{2}m\omega_0^2 r^2 + \frac{1}{2}qB\,\underbrace{(\vec{r}\times\vec{v})_z}_{= r^2\dot{\varphi}}$

$= \frac{1}{2}m(\dot{r}^2 + r^2(\dot{\varphi}^2 + \omega_B\dot{\varphi} - \omega_0^2))$ $\qquad \omega_B = \frac{qB}{m}$

b) φ cyclic : $\frac{\partial L}{\partial \varphi} = 0$

$\Rightarrow \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} \equiv \ell$ constant

$\ell = mr^2(\dot{\varphi} + \frac{1}{2}\omega_B) \quad \Rightarrow \quad \dot{\varphi} = \frac{\ell}{mr^2} - \frac{1}{2}\omega_B$

Physical interpretation : angular momentum, mechanical + electromagnetic contributions

Second const. of motion :

$\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad \frac{dH}{dt} = 0$

$H = p_r\dot{r} + p_\varphi\dot{\varphi} - L = \frac{1}{2}m(\dot{r}^2 + r^2(\dot{\varphi}^2 + \omega_0^2)) = T + V$

energy conserved

c) $\frac{\partial L}{\partial \dot r} = m\dot r \;,\; \frac{\partial L}{\partial r} = mr(\dot\varphi^2 + \omega_B\dot\varphi - \omega_0^2)$

Lagrange's equation $\Rightarrow$

$$\underline{\ddot r - r(\dot\varphi^2 + \omega_B\dot\varphi - \omega_0^2) = 0}$$

$$l^2 = m^2 r^4 (\dot\varphi^2 + \omega_B\dot\varphi + \tfrac{1}{4}\omega_B^2)$$

$$\Rightarrow \quad \underline{\ddot r - \frac{l^2}{m^2 r^3} + r(\omega_0^2 + \tfrac{1}{4}\omega_B^2) = 0}$$

d) Circular motion: $r = \text{const} \Rightarrow \ddot r = 0$

$$\Rightarrow \quad r^4 = \frac{l^2}{m^2\Omega^2} \qquad \Omega \equiv \sqrt{\omega_0^2 + \tfrac{1}{4}\omega_B^2}$$

$$\Rightarrow \quad \underline{r = \sqrt{\frac{l}{m\Omega}}}$$ radius of circle as function of l $(l \neq 0)$

Angular velocity

$$\dot\varphi = \frac{l}{mr^2} - \tfrac{1}{2}\omega_B = \Omega - \tfrac{1}{2}\omega_B = \underline{\sqrt{\omega_0^2 + \tfrac{1}{4}\omega_B^2} - \tfrac{1}{2}\omega_B}$$

If $l = 0$:

$$\ddot r + \Omega^2 r = 0$$ harmonic osc. eq

$$\Rightarrow \quad \underline{r = r_0 \cos\Omega t}$$ r_0 amplitude of radial oscillations

Angle φ of direction of oscillations

$$\dot\varphi = -\tfrac{1}{2}\omega_B \quad \Rightarrow \quad \underline{\varphi = \varphi_0 - \tfrac{1}{2}\omega_B t}$$

rotation with ang. frequency $-\tfrac{1}{2}\omega_B$

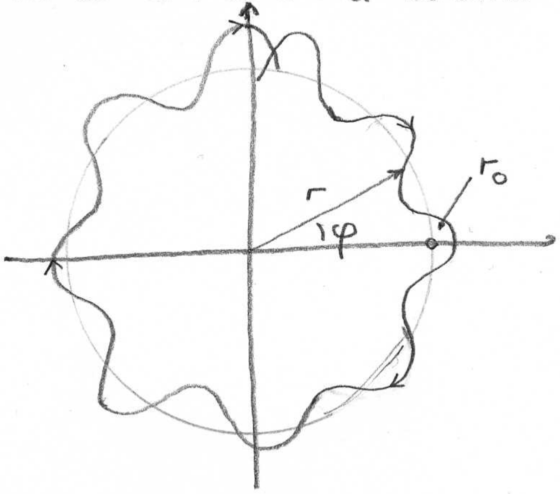
φ_0, r_0 determined by initial conditions

Typical motion ($\ell \neq 0$):

Radial oscillations about equilibrium value $r_0 = \sqrt{\frac{\ell}{m\Omega}}$,

combined with angular velocity $\dot{\varphi} = \frac{\ell}{mr^2} - \frac{1}{2}\omega_B$

$\Rightarrow$ perturbation around circular motion



Problem 2

a) Eq. of motion

$$\dot{\vec{p}} = e\vec{v} \times \vec{B} \; ; \quad \vec{p} = \gamma m \vec{v}$$

$$\dot{\vec{p}} \cdot \vec{v} = 0 \Rightarrow v, \gamma \text{ constant}$$

$$\gamma m \vec{a} = e\vec{v} \times \vec{B} \Rightarrow \gamma m a = e v B$$

$\vec{B}$ const. $\Rightarrow$ circular motion with const. angular velocity

$$v = \omega R, \quad a = \omega^2 R = \frac{v^2}{R}$$

This gives $\gamma m \frac{v^2}{R} = e v B \Rightarrow eB = \frac{\gamma m v}{R} = \underline{\frac{p}{R}}$

b) Energy

$$E^2 = p^2c^2 + m^2c^4 = \gamma^2 m^2 c^4$$

$$\Rightarrow \gamma^2 = \frac{p^2}{m^2c^2} + 1$$

$$\gamma = \sqrt{\frac{p^2}{m^2c^2} + 1} \simeq \frac{p}{mc} = \frac{7 \cdot 10^{12}}{938 \cdot 10^6} = \underline{7460}$$

Note: $\gamma \gg 1 \Rightarrow 1 - \beta^2 \ll 1$ $\quad v$ very close to c.

Acceleration in lab frame

$$a = \frac{v^2}{R} \simeq \frac{c^2}{R} = \frac{9 \cdot 10^8}{2804} \, m/s^2 = \underline{3.2 \cdot 10^{13} \, m/s^2}$$

c) Transformation of fields to instantaneous rest frame:

Since $\vec{v} \cdot \vec{B} = 0$, $\vec{E} = 0$ in lab frame, the general formulas are simplified to

$$\vec{B}' = \gamma \vec{B}, \quad \vec{E}' = \gamma \vec{v} \times \vec{B}$$

Field strengths

$B' = \gamma B = 7460 \times 8.33\,T = \underline{62164\,T}$

orientation: orthogonal to the plane of motion

$E' = \gamma v B \approx v B' = 3.0 \cdot 10^8 \cdot 62164\ T\,m/s = \underline{1.86 \cdot 10^{13}\ V/m}$

orientation: in the plane of motion, orthogonal to the orbit

Proper acceleration = acceleration in the instantaneous rest frame

$m\vec{a}_0 = e\vec{E}' \qquad (v' = 0 \Rightarrow \gamma' = 1)$

$\underline{= \gamma e \vec{v} \times \vec{B}}$

Compare with acceleration in lab frame

$\gamma m \vec{a} = e \vec{v} \times \vec{B}$

$\Rightarrow \vec{a}_0 = \gamma^2 \vec{a}, \qquad a_0 = 7460^2 \cdot 3.2 \cdot 10^{13}\ m/s^2 = \underline{1.79 \cdot 10^{21}\ m/s^2}$

Problem 3

a) Electric dipole moment

$$\vec{p} = \int_{-a/2}^{a/2} z\,\lambda(z,t)\,dz\,\vec{k}$$

$$\Rightarrow \quad \dot{\vec{p}} = \int_{-a/2}^{a/2} z \frac{\partial \lambda}{\partial t}\,dz\,\vec{k}$$

$$= -\int_{-a/2}^{a/2} z \frac{\partial I_1}{\partial z}\,dz\,\vec{k}$$

$$= \int_{-a/2}^{a/2} I_1(z,t)\,dz\,\vec{k} - \underbrace{\left[z I_1\right]_{-a/2}^{a/2}}_{=0}\vec{k}$$

$$= \frac{a}{\pi} I_0 \sin\omega t \left[\sin\frac{\pi z}{a}\right]_{-a/2}^{a/2}\vec{k} = \underline{\frac{2a}{\pi} I_0 \sin\omega t\,\vec{k}}$$

Magnetic dipole moment

$$\vec{m} = \frac{1}{2}\oint \vec{r}\times I_2\,d\vec{r} = \frac{1}{2} I_0 \sin\omega t\,(2a)^2 \int_0^{2\pi} d\varphi\,\vec{k}$$

$$= \underline{4\pi a^2 I_0 \sin\omega t\,\vec{k}}$$

$$\Rightarrow \ddot{\vec{p}} = \frac{2a}{\pi}\omega I_0 \cos\omega t\,\vec{k}\,, \quad \ddot{\vec{m}} = -4\pi a^2\omega^2 I_0 \sin\omega t\,\vec{k}$$

b) Define $\vec{k}\times\vec{n} = \sin\theta\,\vec{e}_1\,,\ (\vec{k}\times\vec{n})\times\vec{n} = \sin\theta\,\vec{e}_2$

$$\Rightarrow \vec{e}_1\cdot\vec{e}_2 = 0,\ \vec{e}_1\cdot\vec{n} = \vec{e}_2\cdot\vec{n} = 0\,,\ \vec{e}_1^{\,2} = \vec{e}_2^{\,2} = 1$$

Poynting's vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}\vec{E}^2\vec{n}$$

$$= \frac{\mu_0}{16\pi^2 r^2 c}\left(\ddot{\vec{p}}^{\,2}\sin^2\theta\,\vec{e}_1^{\,2} + \frac{1}{c^2}\ddot{\vec{m}}^{\,2}\sin^2\theta\,\vec{e}_2^{\,2} - \frac{2}{c}\ddot{\vec{p}}\cdot\ddot{\vec{m}}\sin^2\theta\,\vec{e}_1\cdot\vec{e}_2\right)\vec{n}$$

$$= \frac{\mu_0}{16\pi^2 r^2 c}\left(\ddot{\vec{p}}^{\,2} + \frac{1}{c^2}\ddot{\vec{m}}^{\,2}\right)\sin^2\theta$$

Equal time averages

$$\overline{\ddot{\vec{p}}^{\,2}} = \frac{1}{c^2}\overline{\ddot{\vec{m}}^{\,2}} \Rightarrow \frac{2}{\pi^2}(a\omega)^2 I_0^2 = 8\pi^2 \frac{(a\omega)^4}{c^2} I_0^2$$

$$\Rightarrow \quad a\omega = \frac{c}{2\pi^2}$$

Poynting's vector

$$\vec{S} = \frac{\mu_0}{16\pi^2 r^2 c}\frac{4}{\pi^2}\left(\frac{c}{2\pi^2}\right)^2 I_0^2 \sin^2\theta\, \vec{n}$$

$$= \frac{\mu_0}{16\pi^8 r^2} c\, I_0^2 \sin^2\theta\, \vec{n}$$

Power per unit solid angle in direction $\vec{n}$

$$\frac{dP}{d\Omega} = r^2\vec{S}\cdot\vec{n} = \underline{\frac{\mu_0 c}{16\pi^8} I_0^2 \sin^2\theta} \quad \text{time independent}$$

c) Electric field

$$\underline{\vec{E}(\vec{r},t) = \frac{\mu_0 c}{4\pi^4 r} I_0 (\sin\omega t\, \vec{e}_1 + \cos\omega t\, \vec{e}_2)}$$

$\vec{E}$ rotates in the plane spanned by $\vec{e}_1$ and $\vec{e}_2$:
Circular polarization, lefthanded (clockwise around $\vec{n}$)

If $a\omega \neq \frac{c}{2\pi^2}$: Different coefficients before $\sin\omega t\, \vec{e}_1$ and $\cos\omega t\, \vec{e}_2 \Rightarrow$ elliptic polarization with $\vec{e}_1$ and $\vec{e}_2$ def. symmetry axes

d) Circular receiver antenna: Current induced by e.m.f.

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{r} = -\frac{d\Phi_m}{dt} \quad (\Phi_m \text{ magnetic flux through the loop})$$

Max oscillation amplitude of Φ_m, when the antenna oriented with $\vec{n}$ in the plane of the antenna.