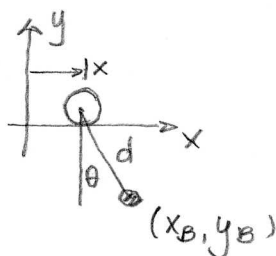


Exam Fys 3120, 2015

Solutions

Problem 1



$$a) \quad x_B = x + d \sin \theta$$

$$y_B = -d \cos \theta$$

$$\dot{x}_B = \dot{x} + d \cos \theta \dot{\theta}, \quad \dot{y}_B = -d \sin \theta \dot{\theta}$$

$$\text{Rolling cylinder: } \dot{x} = \omega R, \quad I = \frac{1}{2} m R^2 \Rightarrow \frac{1}{2} I \omega^2 = \frac{1}{4} m \dot{x}^2$$

Kinetic energy

$$T_A = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \omega^2 = \frac{3}{4} m \dot{x}^2$$

$$T_B = \frac{1}{2} m (\dot{x}_B^2 + \dot{y}_B^2) = \frac{1}{2} m (\dot{x}^2 + d^2 \dot{\theta}^2 + 2d\dot{x}\dot{\theta} \cos \theta)$$

Potential energy

$$V = mgy_B = -mgd \cos \theta$$

Lagrangian

$$L = T - V = \underline{m \left(\frac{5}{4} \dot{x}^2 + \frac{1}{2} d^2 \dot{\theta}^2 + d \cos \theta \dot{x} \dot{\theta} + gd \cos \theta \right)}$$

Constants of motion

$$\frac{\partial L}{\partial x} = 0 \quad (x \text{ cyclic}) \Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m \left(\frac{5}{2} \dot{x} + d \cos \theta \dot{\theta} \right) \text{ constant}$$

$$\text{initial condition} \Rightarrow p_x = 0 \Rightarrow \underline{\dot{x} = -\frac{2}{5} d \cos \theta \dot{\theta}}$$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0 \quad (\text{energy conserved})$$

$$H = \dot{x} p_x + \dot{\theta} p_\theta - L = m \left(\frac{5}{4} \dot{x}^2 + \frac{1}{2} d^2 \dot{\theta}^2 + d \cos \theta \dot{x} \dot{\theta} - gd \cos \theta \right) = T + V$$

$$\text{initial condition} \quad H = E = \underline{-mgd \cos \theta_0}$$

b) Lagrange's equations

$$1) \frac{\partial L}{\partial \dot{x}} = \text{const} \Rightarrow p_x = 0 \quad \underline{\dot{x} = -\frac{2}{5} d \cos \theta \dot{\theta}}$$

$$2) \frac{\partial L}{\partial \theta} = m(d^2 \ddot{\theta} + d \cos \theta \dot{x}) = md^2(1 - \frac{2}{5} \cos^2 \theta) \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -md(\sin \theta \dot{x} \dot{\theta} + g \sin \theta) = md^2(\frac{2}{5} \cos \theta \sin \theta \dot{\theta}^2 - \frac{g}{d} \sin \theta)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow md^2(1 - \frac{2}{5} \cos^2 \theta) \ddot{\theta} + \frac{2}{5} md^2 \cos \theta \sin \theta \dot{\theta}^2 + mgd \sin \theta = 0$$

$$\Rightarrow \underline{(1 - \frac{2}{5} \cos^2 \theta) \ddot{\theta} + \frac{2}{5} \cos \theta \sin \theta \dot{\theta}^2 + \frac{g}{d} \sin \theta = 0}$$

c) Small oscillations $\theta_0 \ll 1$

approximations, keep only first order terms in θ and $\dot{\theta}$,

$$\cos \theta \approx 1, \sin \theta \approx \theta, \dot{\theta}^2 \approx 0 \Rightarrow$$

$$\frac{3}{5} \ddot{\theta} + \frac{g}{d} \theta = 0, \quad \underline{\ddot{\theta} + \frac{5g}{3d} \theta = 0}$$

harmonic oscillator of the form $\ddot{\theta} + \omega^2 \theta = 0$

with angular frequency $\omega = \sqrt{\frac{5g}{3d}}$, solution $\underline{\theta = \theta_0 \cos \omega t}$

Corresponding expression for x

$$\dot{x} \approx -\frac{2}{5} d \dot{\theta} \Rightarrow x = -\frac{2}{5} d \theta + x_0$$

$$\text{initial condition} \Rightarrow x_0 = \frac{2}{5} d \theta_0$$

$$\Rightarrow \underline{x = \frac{2}{5} d \theta_0 (1 - \cos \omega t)}$$

Problem 2

a) Conservation of relativistic energy and momentum in S'

$$\text{I} \quad m_{\tau} c^2 = E_{\nu}' + E_{\kappa}'$$

$$\text{II} \quad \vec{p}_{\nu}' + \vec{p}_{\kappa}' = 0 \Rightarrow |\vec{p}_{\nu}'| = |\vec{p}_{\kappa}'| \equiv p'$$

Energy momentum relations

$$E_{\kappa}'^2 = c^2 p'^2 + m_{\kappa}^2 c^4$$

$$\text{III} \quad E_{\nu}'^2 = c^2 p'^2 = E_{\kappa}'^2 - m_{\kappa}^2 c^4$$

$$\text{I} \Rightarrow E_{\nu}'^2 = (m_{\tau} c^2 - E_{\kappa}')^2 = E_{\kappa}'^2 - 2E_{\kappa}' m_{\tau} c^2 + m_{\tau}^2 c^4$$

Combined with III :

$$m_{\tau}^2 c^4 - 2E_{\kappa}' m_{\tau} c^2 + m_{\kappa}^2 c^4 = 0$$

$$\Rightarrow E_{\kappa}' = \frac{m_{\tau}^2 + m_{\kappa}^2}{2m_{\tau}} c^2 = \underline{957 \text{ MeV}}$$

$$\Rightarrow E_{\nu}' = m_{\tau} c^2 - E_{\kappa}' = \frac{m_{\tau}^2 - m_{\kappa}^2}{2m_{\tau}} c^2 = \underline{820 \text{ MeV}}$$

$$p' = \frac{E_{\nu}'}{c} = \frac{m_{\tau}^2 - m_{\kappa}^2}{2m_{\tau}} c = \underline{820 \text{ MeV}/c}$$

In S' the situation is rotationally symmetric, no preferred direction for \vec{p}_{ν}' .

b) Velocity of τ^- in ref. frame S

$$\sigma = 0.9c \Rightarrow \beta = 0.9, \gamma = 2.30$$

$$\theta_{\nu}' = \frac{\pi}{2} \Rightarrow \theta_{\kappa}' = -\frac{\pi}{2}$$

$$\Rightarrow \text{in } S' \text{ angles } p_{\nu x}' = 0, p_{\nu y}' = p'; p_{\kappa x}' = 0, p_{\kappa y}' = -p'$$

Lorentz-transformation from S' to S

$$E_\nu = \gamma (E'_\nu + \beta p'_{\nu x} c) = \gamma E'_\nu = \underline{1881 \text{ MeV}}$$

$$E_\kappa = \gamma (E'_\kappa + \beta p'_{\kappa x} c) = \gamma E'_\kappa = \underline{2196 \text{ MeV}}$$

$$p_{\nu x} = \gamma (p'_{\nu x} c + \beta E'_\nu) / c = \gamma \beta E'_\nu / c = \underline{1693 \text{ MeV}/c}$$

$$p_{\nu y} = p'_{\nu y} = p' = \underline{820 \text{ MeV}/c}$$

$$p_{\kappa x} = \gamma (p'_{\kappa x} c + \beta E'_\kappa) / c = \gamma \beta E'_\kappa / c = \underline{1976 \text{ MeV}/c}$$

$$p_{\kappa y} = p'_{\kappa y} = -\underline{820 \text{ MeV}/c}$$

Angles

$$\theta_\nu = \text{Arctan} \left(\frac{p_{\nu y}}{p_{\nu x}} \right) = \underline{25.8^\circ}$$

$$\theta_\kappa = \text{Arctan} \left(\frac{p_{\kappa y}}{p_{\kappa x}} \right) = \underline{-22.5^\circ}$$

c) All directions in S' are equally probable, which means that in this frame there is 50% probability to find the direction of the neutrino with $|\theta'_\nu| < \pi/2$.

In S this corresponds to the condition $|\theta_\nu| < 25.8^\circ$.

The energy in S will increase with smaller angle $|\theta_\nu|$, which means larger positive component $p_{\nu x}$.

Since 50% have angles $|\theta_\nu| < 25.8^\circ$, the same percentage will have energy $E_\nu > 1881 \text{ MeV}$, which is the energy for $|\theta_\nu| = 25.8^\circ$.

Problem 3

a) Plane wave $\vec{B}(\vec{r}, t) = \frac{1}{c} \vec{n} \times \vec{E}(\vec{r}, t)$; $\vec{n} = \vec{k}$

$$\Rightarrow B_y = \frac{1}{c} E_x = \frac{E_0}{c} \cos(kz - \omega t), \quad B_x = B_z = 0$$

$$\text{Poynting's vector } \vec{S}_{pw} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} E_x^2 \vec{k} = \frac{E_0^2}{\mu_0 c} \cos^2(kz - \omega t) \vec{k}$$

b) Assume

$$\dot{x} = -\frac{eE_0}{m\omega} \sin(kz - \omega t), \quad \dot{y} = \dot{z} = 0$$

for the coordinates of the electron

$$\Rightarrow \ddot{x} = \frac{eE_0}{m} \cos(kz - \omega t), \quad \ddot{y} = \ddot{z} = 0 \Rightarrow \underline{m\vec{a} = e\vec{E}}$$

Correct only if we can neglect magnetic force $e\vec{v} \times \vec{B}$ and assume non-relativistic motion.

Magnetic force: $e\vec{v} \times \vec{B} = e\dot{x} B_y \vec{k} = e \frac{\dot{x}}{c} E_x \vec{k}$, negligible if

$$|\vec{v} \times \vec{B}| \ll |\vec{E}| \Rightarrow \frac{|\dot{x}|}{c} \ll 1 \quad \text{i.e. non-relativistic speed}$$

$$\text{satisfied if } \underline{\frac{eE_0}{m\omega c} \ll 1}$$

c) Time averaged emitted power from the electron

$$\bar{P} = \frac{\mu_0 e^2}{6\pi c} \overline{\vec{a}^2} \quad (\text{Larmor's formula})$$

$$= \frac{\mu_0 e^4}{6\pi m^2 c} \overline{E^2} = \frac{\mu_0 e^4}{12\pi m^2 c} E_0^2$$

Energy current density of the plane wave, time averaged

$$\bar{S}_{pw} = \frac{1}{\mu_0 c} \overline{E^2} = \frac{1}{2\mu_0 c} E_0^2$$

$$\text{Interaction cross section } \sigma = \frac{\bar{P}}{\bar{S}_{pw}} = \frac{\mu_0^2 e^4}{6\pi m^2}$$

d) Assume electron oscillates about $x=y=z=0$

Differential power in the radial direction

$$\frac{dP}{d\Omega} = r^2 \vec{S}_{\text{rad}} \cdot \vec{n}, \quad \vec{n} = \frac{\vec{r}}{r} \quad \vec{S}_{\text{rad}} = \text{Poynting's vector of the radiation}$$

Assume \vec{r} far from the electron

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0} \vec{E}_{\text{rad}} \times \vec{B}_{\text{rad}}$$

$$= \frac{c}{\mu_0} B_{\text{rad}}^2 \vec{n}$$

$$= \frac{c}{\mu_0} \frac{\mu_0^2 e^2}{16\pi^2 c^2} \left(\frac{\vec{a} \times \vec{n}}{r} \right)_{\text{ret}}^2 \vec{n}$$

$$= \frac{\mu_0 e^2}{16\pi^2 c r^2} (\vec{a}^2 - (\vec{a} \cdot \vec{n})^2)_{\text{ret}} \vec{n}$$

Introduce angle θ between \vec{n} and x -axis

$$\Rightarrow \vec{a}^2 - (\vec{a} \cdot \vec{n})^2 = a^2 (1 - \cos^2 \theta) = \frac{e^2 E_0^2}{m^2} \cos^2(kz - \omega t_{\text{ret}}) \sin^2 \theta$$

time averaged power

$$\frac{d\bar{P}}{d\Omega} = r^2 \overline{\vec{S}_{\text{rad}} \cdot \vec{n}} = \frac{\mu_0 e^4 E_0^2}{32\pi^2 c m^2} \sin^2 \theta$$

maximal for $\theta = \frac{\pi}{2}$, directions orthogonal to x -axis

minimal for $\theta = 0, \pi$ — " — along the — " —

Polarization:

$$\vec{E}_{\text{rad}}(\vec{r}, t) = \frac{\mu_0 e^2 E_0}{4\pi m} \cos(kz - \omega t_{\text{ret}}) (\vec{i} \times \vec{n}) \times \vec{n}$$

Linear polarization in direction $(\vec{n} \times \vec{i}) \times \vec{n} = \underline{\vec{i} - (\vec{n} \cdot \vec{i}) \vec{n}}$