Exam FYS3120, spring semester 2016
Solutions

PROBLEM 1

a) Coordinates of the tube

$$
\begin{aligned}
x_t = s\cos\alpha \quad &\Rightarrow \quad \dot{x}_t = \dot{s}\cos\alpha \\
y_t = -s\sin\alpha \quad &\Rightarrow \quad \dot{y}_t = -\dot{s}\sin\alpha
\end{aligned} \tag{1}
$$

coordinates of the ball

$$
\begin{aligned}
x_b = x_t + b\sin\theta \quad &\Rightarrow \quad \dot{x}_t = \dot{s}\cos\alpha + b\dot{\theta}\cos\theta \\
y_b = y_t - b\cos\theta \quad &\Rightarrow \quad \dot{y}_t = -\dot{s}\sin\alpha + b\dot{\theta}\sin\theta
\end{aligned} \tag{2}
$$

Kinetic energy

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x}_t^2 + \dot{y}_t^2) + \frac{1}{2}m(\dot{x}_b^2 + \dot{y}_b^2) \\
&= \frac{1}{2}\dot{s}^2 + \frac{1}{2}m(\dot{s}^2 + b^2\dot{\theta}^2 + 2b\dot{s}\dot{\theta}(\cos\alpha\cos\theta - \sin\alpha\sin\theta)) \\
&= \dot{s}^2 + \frac{1}{2}mb^2\dot{\theta}^2 + mb\dot{s}\dot{\theta}\cos(\theta+\alpha)
\end{aligned} \tag{3}
$$

Potential energy

$$
V = mg(y_t + y_b) = -2mgs\sin\alpha - mgb\cos\theta \tag{4}
$$

Lagrangian

$$
\underline{L = T - V = m\dot{s}^2 + \frac{1}{2}mb^2\dot{\theta}^2 + mb\dot{s}\dot{\theta}\cos(\theta+\alpha) + 2mgs\sin\alpha + mgb\cos\theta} \tag{5}
$$

b) Lagrange's equations,

$$
\begin{aligned}
&\frac{\partial L}{\partial \dot{s}} = 2m\dot{s} + mb\dot{\theta}\cos(\theta+\alpha)\,, \quad \frac{\partial L}{\partial s} = 2mg\sin\alpha \\
&\Rightarrow \underline{\ddot{s} + \frac{1}{2}b\ddot{\theta}\cos(\theta+\alpha) - \frac{1}{2}b\dot{\theta}^2\sin(\theta+\alpha) - g\sin\alpha = 0}
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&\frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} + 2mb\dot{s}\cos(\theta+\alpha)\,, \quad \frac{\partial L}{\partial \theta} = -mb\dot{s}\dot{\theta}\sin(\theta+\alpha) - mgb\sin\theta \\
&\Rightarrow b\ddot{\theta} + \ddot{s}\cos(\theta+\alpha) - \dot{s}\dot{\theta}\sin(\theta+\alpha) + \dot{s}\dot{\theta}\sin(\theta+\alpha) + g\sin\theta = 0 \\
&\Rightarrow \quad \underline{b\ddot{\theta} + \ddot{s}\cos(\theta+\alpha) + g\sin\theta = 0}
\end{aligned} \tag{7}
$$

c) Eliminate s by subtracting equations, (7) - (6) $\cos(\theta+\alpha)$:

$$
\begin{aligned}
&b\ddot{\theta}(1 - \frac{1}{2}\cos^2(\theta+\alpha)) + \frac{1}{2}b\dot{\theta}^2\sin(\theta+\alpha)\cos(\theta+\alpha) + g(\sin\theta + \cos(\theta+\alpha)\sin\alpha) = 0 \\
&\Rightarrow \quad \underline{b\ddot{\theta}(1 - \frac{1}{2}\cos^2(\theta+\alpha)) + \frac{1}{2}b\dot{\theta}^2\sin(\theta+\alpha)\cos(\theta+\alpha) + g\sin(\theta+\alpha)\cos\alpha = 0}
\end{aligned} \tag{8}
$$

Assume constant value $\theta = \theta_0 \Rightarrow \ddot{\theta} = \dot{\theta} = 0$. Inserted in Eq.(8), the terms involving time derivatives vanish, and the last term also vanishes provided

$$\sin(\theta + \alpha) = 0 \quad \Rightarrow \underline{\theta = -\alpha} \tag{9}$$

d) Small deviations from $\theta = -\alpha$,

$$\theta = -\alpha + \eta, \quad \dot{\theta} = \dot{\eta}, \quad \ddot{\theta} = \ddot{\eta} \tag{10}$$

Eq.(8) expanded to first order in η and its time derivatives,

$$\frac{1}{2} b\ddot{\eta} + g\cos\alpha\, \eta = 0 \tag{11}$$

This is a harmonic oscillator equation with solution

$$\eta(t) = \eta_0 \cos(\omega t + \phi) \tag{12}$$

and angular frequency $\underline{\omega = \sqrt{\frac{2g}{b}\cos\alpha}}$.

PROBLEM 2

a) The velocity v of the electron in the lab frame S,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \Rightarrow \quad v = \sqrt{1 - \frac{1}{\gamma^2}}\, c = \underline{0.99995c} \tag{13}$$

Angular frequency,

$$\omega = \frac{v}{R} \approx \frac{c}{R} = \underline{3.0 \cdot 10^7 s^{-1}} \tag{14}$$

Acceleration

$$a = \frac{v^2}{R} = \omega^2 R = \underline{9.0 \cdot 10^{15} m/s^2} \tag{15}$$

Particle coordinates

$$x^0 = ct\,, \quad x^1 = x = R\sin\omega t\,, \quad x^2 = y = -R\cos\omega t \tag{16}$$

Proper time (time dilatation formula): $\tau = t/\gamma \quad \Rightarrow$

$$x^0(\tau) = \gamma c\tau\,, \quad x^1(\tau) = x = R\sin(\gamma\omega\tau)\,, \quad x^2 = y = -R\cos(\gamma\omega\tau) \tag{17}$$

b) Four-velocity,

$$U^\mu = \frac{dx^\mu}{d\tau} \quad \Rightarrow \quad U^0 = \gamma c\,, \quad U^1 = R\gamma\omega\cos(\gamma\omega\tau)\,, \quad U^2 = R\gamma\omega\sin(\gamma\omega\tau) \tag{18}$$

Four-acceleration,

$$A^\mu = \frac{dU^\mu}{d\tau} \quad \Rightarrow \quad A^0 = 0, \quad A^1 = -R\gamma^2\omega^2\sin(\gamma\omega\tau)\,, \quad A^2 = R\gamma^2\omega^2\cos(\gamma\omega\tau) \tag{19}$$

Proper acceleration = acceleration in the instantaneous inertial rest frame:

$$(A^\mu) = (0, \mathbf{a_0}) \quad \Rightarrow \quad a_0^2 = -A_\mu A^\mu = R^2\gamma^4\omega^4$$
$$\Rightarrow a_0 = \gamma^2 R\omega^2 = 10^4 a = \underline{9.0 \cdot 10^{19} m/s^2} \tag{20}$$

c) The instantaneous inertial rest frame is an inertial frame where the particle has velocity $v = 0$ at a given instant. S' is the instantaneous inertial rest frame of the particle at the instant where the particle is at point A. The coordinates $(0, 0, 0)$ in S' corresponds to the coordinates $(0, 0, -R)$ in S.

Lorentz transformation from S to S',

$$\begin{aligned} t' &= \gamma(t - \frac{v}{c^2}x) \\ x' &= \gamma(x - vt) \\ y' &= y + R \end{aligned} \tag{21}$$

If $t' = 0$:

$$t = \frac{v}{c^2}x \quad \Rightarrow \quad x' = \gamma(1 - \frac{v^2}{c^2})x = \frac{1}{\gamma}x \tag{22}$$

The transformation of the circle in S, for $t' = 0$

$$\begin{aligned} & x^2 + y^2 = R^2 \\ \Rightarrow \quad & \gamma^2 x'^2 + (y' - R)^2 = R^2 \\ \Rightarrow \quad & \frac{(x' - x_0')^2}{a^2} + \frac{(y' - y_0')^2}{b^2} = 0 \end{aligned} \tag{23}$$

with $x_0' = 0, y_0' = R, a = R/\gamma, b = R$.

Eq.(23) shows that the ring takes the form of an ellipse in S', with short axis $a = R/\gamma$ and long axis $b = R$. This is consistent with length contraction in the x-direction.

PROBLEM 3

a) Endpoint charges: $\pm Q(t),\ Q(0) = 0$. Charge conservation,

$$\begin{aligned} & \frac{dQ}{dt} = I = I_0 \sin\omega t \\ \Rightarrow \quad & Q(t) = \int_0^t I_0 \sin\omega t\, dt = \left[-\frac{I_0}{\omega}\cos\omega t\right]_0^t = \frac{I_0}{\omega}(1 - \cos\omega t) \end{aligned} \tag{24}$$

Electric dipole moment

$$\begin{aligned} & \mathbf{p}(t) = 2Q(t)a\,\mathbf{k} = \underline{\frac{2aI_0}{\omega}(1 - \cos\omega t)\mathbf{k}} \\ \Rightarrow \quad & \ddot{\mathbf{p}} = 2I_0\omega a \cos\omega t\mathbf{k} \end{aligned} \tag{25}$$

b) Electic dipole radiation, magnetic field

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi rc}\ddot{\mathbf{p}}_{ret} \times \mathbf{n} \\ &= \frac{\mu_0 I_0\omega a}{2\pi rc}\cos\omega t_r\,\mathbf{k} \times \mathbf{e}_r \\ &= \underline{\frac{\mu_0 I_0\omega a}{2\pi rc}\sin\theta\cos\omega t_r\,\mathbf{e}_\phi} \qquad t_r = t - \frac{r}{c} \end{aligned} \tag{26}$$

Electric field

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= c\mathbf{B}(\mathbf{r},t)\times\mathbf{e}_r \\ &= \underline{\frac{\mu_0 I_0 \omega a}{2\pi r}\sin\theta\cos\omega t_r\,\mathbf{e}_\theta} \end{aligned} \tag{27}$$

The radiation is linearly polarized. $\mathbf{E}$ oscillates with time in the fixed direction defined by $\mathbf{e}_\theta$. Similarly $\mathbf{B}$ oscillates in the orthogonal direction, given by $\mathbf{e}_\phi$.

c) Write the fields as $\mathbf{B} = B\mathbf{e}_\phi$ and $\mathbf{E} = cB\mathbf{e}_\theta$, Poynting's vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{c}{\mu_0}B^2\mathbf{e}_\theta\times\mathbf{e}_\phi = \frac{c}{\mu_0}B^2\mathbf{e}_r = \frac{\mu_0\omega^2 a^2}{4\pi^2 r^2 c}I_0^2\sin^2\theta\cos\omega t_r\,\mathbf{e}_r \tag{28}$$

Radiated power per unit solid angle

$$\frac{dP}{d\Omega} = r^2 S = \frac{c}{\mu_0}r^2B^2 = \frac{\mu_0\omega^2a^2}{4\pi^2 c}I_0^2\sin^2\theta\cos\omega t_r \tag{29}$$

Averaged over time

$$\overline{\frac{dP}{d\Omega}} = \frac{\mu_0\omega^2a^2}{8\pi^2c}I_0^2\sin^2\theta \tag{30}$$

Integrated over angles

$$\begin{aligned} &\int d\Omega\sin^2\theta = 2\pi\int_0^\pi \sin^3\theta d\theta = \frac{8}{3}\pi \\ \Rightarrow\quad &\bar{P} = \frac{\mu_0\omega^2a^2}{3\pi c}I_0^2 \equiv \frac{1}{2}RI_0^2 \\ \Rightarrow\quad &\underline{R = \frac{2\mu_0\omega^2a^2}{3\pi c}} \end{aligned} \tag{31}$$

With R_0 as regular resistance and R as radiation resistance, the total power consumed by the antenna is

$$\bar{P} = \frac{1}{2}(R+R_0)I_0^2 \tag{32}$$

d) Specified values, $2a = 5\,\text{cm}$, frequency $f = 150\,\text{MHz}$ and current $I_0 = 30\,\text{A}$. The radiation resistance and the time averaged radiation power in this case are

$$R = \frac{8\pi\mu_0 f^2 a^2}{3c} = \underline{0.49\,\Omega} \tag{33}$$

$$\bar{P} = \frac{1}{2}RI_0^2 = \underline{222\,W} \tag{34}$$