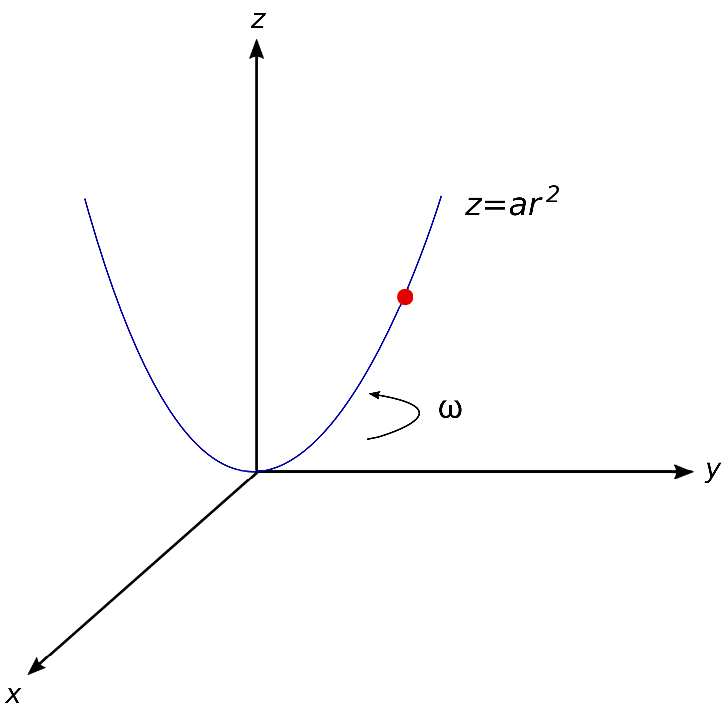
Question 1 Lagrangian mechanics

A particle of mass m moves without friction on a parabola-shaped stiff string under the effects of gravity. The string rotates about the z-axis with constant angular velocity ω, and its shape is given by the equation $z = ar^2$, where a is a constant and r is the distance from the z-axis.

a) Sketch a figure of the system. [2 points]

Answer:



Figur 1: Mass m moving on a parabola-shaped stiff string with constant angular velocity ω.

b) How many degrees of freedom does this system have? [4 points]

Answer: The number of degrees of freedom in three dimensions is $d = 3N - M$, where N is the number of particles and M in the number of constraints. In this case there is $N = 1$ particles, while the string gives two constraints: one that fixes the z-coordinate, and the constant angular velocity fixing the angle ϕ in the plane at any given time as $\phi(t) = \phi_0 + \omega t$, a time-dependent constraint. Thus $d = 3 \cdot 1 - 2 = 1$ and there is one degree of freedom.

c) Find the kinetic energy K of the mass m in terms of r. [5 points]

Answer: The kinetic energy is defined as

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{1}$$

In polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$, and using $z = ar^2$, this is can be written

$$\begin{aligned} K &= \frac{1}{2}m\left[(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)^2 + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)^2 + (2ar\dot{r})^2\right] \\ &= \frac{1}{2}m\left[(1 + 4a^2r^2)\dot{r}^2 + \omega^2 r^2\right]. \end{aligned} \tag{2}$$

d) Show that the Lagrangian of the system can be written

$$L = \frac{1}{2}m\left[(1 + 4a^2r^2)\dot{r}^2 + r^2(\omega^2 - 2ga)\right], \tag{3}$$

where g is the acceleration due to gravity (in the negative z-direction). [3 points]

Answer: The Lagrangian is given as $L = K - V$ so

$$\begin{aligned} L &= K - mgz = \frac{1}{2}m\left[(1 + 4a^2r^2)\dot{r}^2 + \omega^2r^2\right] - mgar^2 \\ &= \frac{1}{2}m\left[(1 + 4a^2r^2)\dot{r}^2 + (\omega^2 - 2ga)r^2\right]. \end{aligned} \tag{4}$$

e) Is the angular momentum around the z-axis a constant of motion / conserved quantity? [2 points]

Answer: No. L does not depend on ϕ, however, ϕ is not a generalized coordinate. Another way to see this is that the angular momentum $\vec{\ell} = m\vec{r} \times \vec{v}$ is not constant since the magnitude (but not direction) of v is constant (constant ω), but r varies with time.

f) Show that the equation of motion for m is

$$(1 + 4a^2r^2)\ddot{r} + 4a^2r\dot{r}^2 - (\omega^2 - 2ga)r = 0. \tag{5}$$

[5 points]

Answer: We find the ingredients in Lagrange's equation

$$\begin{aligned} \frac{\partial L}{\partial r} &= \frac{1}{2}m\left[(8a^2r)\dot{r}^2 + 2(\omega^2 - 2ga)r\right] = 4a^2mr\dot{r}^2 + (\omega^2 - 2ga)mr, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= \frac{d}{dt}\left(m(1 + 4a^2r^2)\dot{r}\right) = 8a^2mr\dot{r}^2 + (1 + 4a^2r^2)m\ddot{r}. \end{aligned} \tag{6}$$

So

$$
\begin{aligned}
\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 0 \\
4a^2 m r \dot{r}^2 + (\omega^2 - 2ga)mr - 8a^2 m r \dot{r}^2 - (1 + 4a^2 r^2) m \ddot{r} &= 0 \\
(1 + 4a^2 r^2) m \ddot{r} + 4a^2 m r \dot{r}^2 - (\omega^2 - 2ga)mr &= 0 \\
(1 + 4a^2 r^2) \ddot{r} + 4a^2 r \dot{r}^2 - (\omega^2 - 2ga)r &= 0. \quad (7)
\end{aligned}
$$

g) Show that there is a solution with constant r, and find the angular velocity ω for this solution. What is the physical interpretation of the solution? [5 points]

Answer: For $\omega = \sqrt{2ga}$, the e.o.m. reduces to

$$
(1 + 4a^2 r^2) \ddot{r} + 4a^2 r \dot{r}^2 = 0. \quad (8)
$$

If r is constant $\dot{r} = 0$ and $\ddot{r} = 0$, which automatically fulfils the e.o.m. The motion of m will be a circular orbit at height $z = ar^2$, where the force of gravity is equal and opposite to the force from the string, caused by the rotation, minus the force needed for the acceleration to remain in circular motion.

Question 2 Relativistic mechanics

First we warm up a little by doing the following:

a) Explain what we mean by a Lorentz vector. (Sometimes sloppily just called a four-vector in the course.) [2 points]

Answer: A Lorentz vector is a four-component vector that transforms the same way as the time and space coordinate vector $x^\mu = (ct, \vec{r})$ under Lorentz transformations.

b) Explain why $p^\mu \equiv mU^\mu$ is a Lorentz vector. Here $U^\mu \equiv \frac{dx^\mu}{d\tau}$, m is the mass of the particle, and τ is the proper time. [3 points]

Answer: Since the proper time is a Lorentz invariant (does not change under Lorentz transformations), and x^μ by definition is a Lorentz vector, then U^μ must also be a Lorentz vector. m is also a Lorentz scalar, so p^μ must be a Lorentz vector.

c) Find $p^2 = p^\mu p_\mu$. [3 points]

Answer: p^2 is a Lorentz invariant so it does not change between reference frames. In the rest frame of the particle $p^\mu = (mc, 0)$, so $p^2 = m^2c^2$.

We will now use this look at the so-called **pair-production** reaction where a photon γ produces an electron–positron pair e^+e^- in an interaction with a charged particle N:

$$\gamma + N \to N + e^+ + e^-. \tag{9}$$

d) Assume that the mass of the charged particle N is M and that the masses of the electron and positron are both m. Find the smallest photon energy E_γ where the reaction is possible in the laboratory reference frame where N is at rest. Discuss the physics of this threshold in the two cases: i) $M \gg m$ and ii) $M/m \to 0$. *Hint:* It can be useful to consider an invariant in two different rest frames for the two sides of the reaction. [10 points]

Answer: Conservation of (relativistic) energy and momentum requires $p_\gamma^\mu + p_N^\mu = p_N^{'\mu} + p_{e+}^\mu + p_{e-}^\mu$. Squaring both sides gives

$$p_\gamma^2 + p_N^2 + 2p_\gamma p_N = p_N^{'2} + p_{e+}^2 + p_{e-}^2 + 2p_N' p_{e^+} + 2p_N' p_{e^-} + 2p_{e^+}p_{e^-}, \tag{10}$$

or in terms of masses

$$M^2c^2 + 2p_\gamma p_N = M^2c^2 + 2m^2c^2 + 2p_N' p_{e^+} + 2p_N' p_{e^-} + 2p_{e^+}p_{e^-}. \tag{11}$$

Keep in mind that both sides of this equation are Lorentz invariants and thus independent of rest frames. If we use that $p_N^\mu = (Mc, 0)$ in the laboratory frame, the left hand side is $M^2c^2 + 2\frac{E_\gamma}{c}Mc$. If we use the centre-of-mass rest frame for the right-hand side $\vec{p}_N' + \vec{p}_{e^+} + \vec{p}_{e^-} = 0$. The minimum amount of kinetic energy that fulfils this requirement — thus giving the minimum photon energy — is if they are all at rest, so

$$\begin{aligned} & M^2c^2 + 2m^2c^2 + 2p_N' p_{e^+} + 2p_N' p_{e^-} + 2p_{e^+}p_{e^-} \\ = \;& M^2c^2 + 2m^2c^2 + 2Mcmc + 2Mcmc + 2m^2c^2 \\ = \;& M^2c^2 + 4m^2c^2 + 4Mmc^2. \end{aligned} \tag{12}$$

Putting these results back into (11) gives

$$E_\gamma = \frac{2m^2c^2 + 2Mmc^2}{M} = 2mc^2(1 + \frac{m}{M}). \tag{13}$$

For case i) $E_\gamma \simeq 2mc^2$, thus if the charged particle is much more massive than the electron the pair-productions is allowed as soon as the energy of the photon is above the rest energy of the electron–positron pair. For ii) $E_\gamma \to \infty$, so for a very light charged particle (or in the vacuum!) the prosess is never possible.

e) The centre-of-mass (CM) system, or reference frame, is defined as the reference frame where the sum of the incomming (and outgoing) particle momenta are zero. Find the velocity of the CM system v_{CM} with respect to the laboratory reference frame as a function of the photon energy E_γ. *Hint:* You do not need the answer from the previous question. [7 points]

Answer: In the CM RF $\vec{p}'_\gamma + \vec{p}'_N = 0$ which means that $E'_\gamma = -cp'_N$. Here we use the prime to indicate that we are not in the laboratory frame. The transformation from the laboratory frame is given by

$$\begin{aligned} E'_\gamma &= \gamma(E_\gamma - v_{CM}p_\gamma) = \gamma\left(1 - \frac{v_{CM}}{c}\right)E_\gamma, \\ p'_N &= \gamma\left(p_N - \frac{v_{CM}}{c^2}E_N\right) = -\gamma v_{CM}M. \end{aligned} \tag{14}$$

Thus

$$\gamma\left(1 - \frac{v_{CM}}{c}\right)E_\gamma = c\gamma v_{CM}M, \tag{15}$$

and

$$v_{CM} = \frac{E_\gamma c}{E_\gamma + Mc^2}. \tag{16}$$

Question 3 Electromagnetism

Assume that we have a monochromatic plane wave of the form

$$\vec{E}(\vec{r},t) = E_0\cos(kz - \omega t)\hat{e}_x. \tag{17}$$

a) Find Poynting's vector $\vec{S}$ for this wave in terms of $\vec{E}$. [5 points]

Answer: Poynting's vector is given as

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\vec{E}\times\left(\frac{1}{c}\hat{n}\times\vec{E}\right) \\ &= \frac{1}{\mu_0}\frac{1}{c}\hat{n}\left(\vec{E}\cdot\vec{E}\right) - \frac{1}{\mu_0}\vec{E}\left(\vec{E}\cdot\frac{1}{c}\hat{n}\right) \\ &= \frac{1}{c\mu_0}E^2\hat{e}_z = c\epsilon_0 E^2\hat{e}_z \end{aligned} \tag{18}$$

where we have used the following expression from Rottmann for the cross product: $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$, and that the unit vector in the direction of propogation is $\hat{n} = \vec{k}/k = \hat{e}_z$.

b) What is the physical interpretation of Poynting's vector? [2 points]

Answer: Poynting's vector gives the energy current density, meaning it gives the amount of energy flowing through a unit area perpendicular to the vector per unit time.

We will now look at the scattering of such as wave on a free electron of mass m and charge e. We assume that the electron is initially at rest.

c) Explain why we can ignore the force from the magnetic field as long as $v \ll c$, where v is the velocity of the electron. [4 points]

Answer: The Lorentz force is given as $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$. The force from the magnetic field is

$$\vec{F}_{mag} = e\vec{v} \times \vec{B} = e\vec{v} \times \left(\frac{1}{c}\hat{n} \times \vec{E}\right) = \frac{v}{c}e\hat{v} \times (\hat{n} \times \vec{E}), \tag{19}$$

with magnitude $|\vec{F}_{mag}| = e\beta E_0$, which is much less than the magnitude of the force from the electric field $|\vec{F}_{el}| = eE_0$, since $\beta \ll 1$.

d) Find the power P radiated by the electron per unit time. Express your answer in terms of the **classical electron radius**

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}. \tag{20}$$

[4 points]

Answer: The power P radiated per unit time for a non-relativistic charged particle with charge e is given by Larmor's formula

$$P(t) = \frac{\mu_0 e^2}{6\pi c}a^2, \tag{21}$$

where a is the acceleration. Here $\vec{F} = m\vec{a} = e\vec{E}$, so that

$$\vec{a} = \frac{e}{m}\vec{E}, \tag{22}$$

and

$$P(t) = \frac{\mu_0 e^4}{6\pi m^2 c}E_0^2 \cos^2(\omega t) = \frac{8\pi}{3}r_0^2 c\epsilon_0 E_0^2 \cos^2(\omega t), \tag{23}$$

where we have placed the electron at the origin $z = 0$.

e) Find the **cross section** σ of the electron for the electromagnetic scattering, given as $\sigma = P/S$, where $S = |\vec{S}|$. Comment on the units of the cross section. [4 points]

Answer:

$$\sigma = \frac{P}{S} = \frac{\frac{8\pi}{3}r_0^2 c\epsilon_0 E_0^2 \cos^2(\omega t)}{c\epsilon_0 E_0^2 \cos^2(\omega t)} = \frac{8\pi}{3}r_0^2, \tag{24}$$

where we have again placed the electron at the origin. As we see the cross section indeed has units of area.[1]

[1] In fact it is used as a measure of the effective size of a particle in interactions (in this case electromagnetic interactions), which explains the (classical) electron radius name for r_0.

Question 4 Ehrenfest's paradox

Imagine that you observing a rigid disc of radius R rotating with a large (relativistic) angular velocity ω. We will use the laboratory rest frame as the reference frame of the problem, with the origin in the centre of the disc. What will you observe for the diameter and the circumference of the disc, and their ratio, using special relativity? [5 points]

Answer: The circumference C of the non-rotating disc is $C = 2\pi R$ wile the diameter D is $D = 2R$. The ratio is naturally $C/R = \pi$. On the rotating disc the distance to the edge is perpendicular to the velocity, thus experiencing no length contraction. The observed diameter is then the same as at rest, $D' = 2R$. An infinitesimal length element along the edge, dC, is along the direction of velocity, thus is contracted as dC/γ. The gamma factor is constant along the circumference, thus the observed circumference is $C' = 2\pi R/\gamma$. The ratio of the two is

$$\frac{C'}{R'} = \frac{\pi}{\gamma} = \pi\sqrt{1 - \frac{v^2}{c^2}} = \pi\sqrt{1 - \frac{(\omega R)^2}{c^2}}. \tag{25}$$

The paradox here is that this seemingly contradicts Euclidean geometry by making π smaller. This problem find its resolution in general relativity, however, it has lead to many productive discussions, see *e.g.* Ø. Grøn, *Relativistic description of a rotating disk*, Amer. J. Phys. 43 (10): 869–876 (1975).