**Question 1 Swinging Lagrangian mechanics**

A mechanical system consists of two pendula with mass $m$ and length $l$ swinging from fixed points a distance $d$ apart. The pendula are coupled by an (effectively weightless) spring with spring constant $k$, and move under the influence of gravity. The pendula swing in the same plane and the spring has an unstretched length $d_0$, *viz.* the length when no spring force is acting. See illustration in Fig. 1.

We remind you that the potential energy for a spring is given by $V = \frac{1}{2}kx^2$, when $x$ is the displacement of the string length.

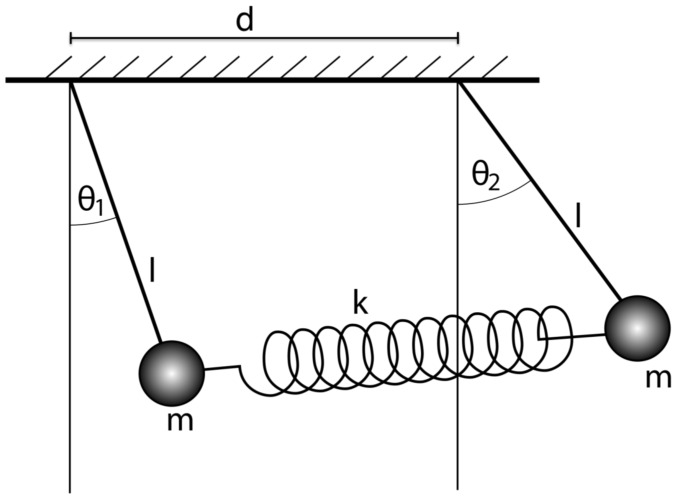


Figure 1: Two coupled pendula.

**a)** How many degrees of freedom does this system have? Explicitly give your choice of generalised coordinates. [3 points]

**Answer:** The two masses each have a set of 2D-coordinates since the pendula swing in a plane. There are two constraints from the length of the pendula. In total this gives $d = 2N - M = 2 \cdot 2 - 2 = 2$ degrees of freedom. In the following we will use the angles, $\theta_1$ and $\theta_2$, of the pendula as generalised coordinates since we are later told these are small. Other choices are possible, *e.g.* the horizontal displacement of the pendula masses.

**b)** Find the potential energy of the system in terms of the generalised coordinates assuming that the angles $\theta_1$ and $\theta_2$ are small. We shall keep to this assumption in the following. *Hint:* the small angle expansions of sine and cosine to second order in angles are

$$\sin\theta = \theta + \mathcal{O}(\theta^3), \quad \cos\theta = 1 - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^4). \tag{1}$$

[5 points]

**Answer:** There are two sources of potential energy: the gravitational potential for the two masses and the energy stored in the spring. These are $V_{\text{gravity}} = -mgl(\cos\theta_1 + \cos\theta_2)$ and $V_{\text{spring}} = \frac{1}{2}k(d' - d_0)^2$, where $d'$ is the distance between the two pendula masses. When $\theta_1$ and $\theta_2$ are small we can approximate $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, so

$$V_{\text{gravity}} = \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) - 2mgl. \qquad (2)$$

In the following we are free to ignore the constant contribution to the potential, $2mgl$, which can be removed by a redefinition of the zero-level. The distance between the pendula masses requires some trigonometry. Pythagoras gives us

$$\begin{aligned} d'^2 &= (d + l\sin\theta_2 - l\sin\theta_1))^2 + (-l\cos\theta_2 + l\cos\theta_1)^2 \\ &= d^2 + 2dl(\sin\theta_2 - \sin\theta_1) + l^2\left[(\sin\theta_2 - \sin\theta_1)^2 + (\cos\theta_1 - \cos\theta_2)^2\right] \\ &= d^2 + 2dl(\sin\theta_2 - \sin\theta_1) + l^2(2 - 2\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2) \\ &= d^2 + 2dl(\sin\theta_2 - \sin\theta_1) + 2l^2(1 - \cos(\theta_2 - \theta_1)) \qquad (3) \end{aligned}$$

where the initial two parenthesis contain the horizontal and vertical distances between the pendula, respectively. Using that for small angles $\sin\theta = \theta$ we get

$$\begin{aligned} d'^2 &\simeq d^2 + 2dl(\theta_2 - \theta_1) + 2l^2(1 - 1 + \frac{1}{2}(\theta_2 - \theta_1)^2) \\ &= d^2 + 2dl(\theta_2 - \theta_1) + l^2(\theta_2 - \theta_1)^2 \\ &= (d + l(\theta_2 - \theta_1))^2, \qquad (4) \end{aligned}$$

so that $d' \simeq d + l(\theta_2 - \theta_1)$. Alternatively, one could insert the small angle approximation immediately, however, then one would have to argue that terms with $\theta^4$ must be dropped in $d'^2$.

The total potential in the small angle approximation is then

$$V = \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) + \frac{1}{2}k(d - d_0 + l(\theta_2 - \theta_1))^2. \qquad (5)$$

**c)** Find the equilibrium position of the pendula. [4 points]

**Answer:** In terms of generalised coordinates $q_i$ the equilibrium is given by $\frac{\partial V}{\partial q_i} = 0$ for all $i$. We have

$$\begin{aligned} \frac{\partial V}{\partial \theta_1} &= mgl\theta_1 - kl(d - d_0 + l(\theta_2 - \theta_1)) = 0 \qquad (6) \\ \frac{\partial V}{\partial \theta_2} &= mgl\theta_2 + kl(d - d_0 + l(\theta_2 - \theta_1)) = 0 \qquad (7) \end{aligned}$$

Adding the two equations gives $mgl\theta_1 + mgl\theta_2 = 0$, which means $\theta_1 + \theta_2 = 0$, telling us that the equilibrium solution is (as expected for equal masses) symmetric around $\theta = 0$ (vertical position). Inserting this in the first equation yields $mgl\theta_1 - kl(d - d_0) = 0$ or

$$\theta_1 = \frac{k(d - d_0)}{mg + 2kl}. \tag{8}$$

**d)** Show that the Lagrangian of the system can be written

$$L = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) - \frac{1}{2}k(d - d_0 + l(\theta_2 - \theta_1))^2, \tag{9}$$

where $g$ is the acceleration due to gravity. [3 points]

**Answer:** The Lagrangian is given as $L = K - V$, where $K$ is the kinetic energy and $V$ is the potential energy. The total kinetic energy of the pendula is

$$K = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2), \tag{10}$$

where $(x_i, y_i)$ are the pendula coordinates. Setting the horizontal as the $x$-direction and the vertical as the $y$-direction we have $x_i = l\sin\theta_i$ and $y_i = -l\cos\theta_i$. This gives $\dot{x}_i = l\dot{\theta}_i\cos\theta_i$ and $\dot{y}_i = l\dot{\theta}_i\sin\theta_i$ so

$$K = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2), \tag{11}$$

and the Lagrangian is

$$L = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) - \frac{1}{2}k(d - d_0 + l(\theta_2 - \theta_1))^2. \tag{12}$$

**e)** Find the equations of motion. [4 points]

**Answer:** The necessary ingredients to Lagrange's equation are

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= -\frac{\partial V}{\partial \theta_1} = -mgl\theta_1 + kl(d - d_0 + l(\theta_2 - \theta_1)) \\
\frac{\partial L}{\partial \theta_2} &= -\frac{\partial V}{\partial \theta_2} = -mgl\theta_2 - kl(d - d_0 + l(\theta_2 - \theta_1)) \\
\frac{\partial L}{\partial \dot{\theta}_1} &= ml^2\dot{\theta}_1 \\
\frac{\partial L}{\partial \dot{\theta}_2} &= ml^2\dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= ml^2\ddot{\theta}_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= ml^2\ddot{\theta}_2.
\end{aligned}$$

This gives the (coupled) equations of motion

$$ml^2\ddot{\theta}_1 + mgl\theta_1 - kl(d - d_0 + l(\theta_2 - \theta_1)) = 0 \tag{13}$$
$$ml^2\ddot{\theta}_2 + mgl\theta_2 + kl(d - d_0 + l(\theta_2 - \theta_1)) = 0 \tag{14}$$

or

$$ml^2\ddot{\theta}_1 + mgl\theta_1 + kl^2(\theta_1 - \theta_2) - kl(d - d_0) = 0 \tag{15}$$
$$ml^2\ddot{\theta}_2 + mgl\theta_2 + kl^2(\theta_2 - \theta_1) + kl(d - d_0) = 0 \tag{16}$$

**f)** If $d = d_0$, show that

$$\theta_i(t) = T_i e^{\pm i\omega t} \tag{17}$$

are solutions and find the *two* different allowed *magnitudes* of the angular frequency $\omega$. Briefly discuss the physical interpretation of these solutions. [6 points]

**Answer:** We show this by insertion into the equations of motion. We have $\dot{\theta}_i = \pm i\omega\theta_i$ and $\ddot{\theta}_i = -\omega^2\theta_i$, which gives

$$-ml^2\omega^2 T_1 + mglT_1 + kl^2(T_1 - T_2) = 0, \tag{18}$$
$$-ml^2\omega^2 T_2 + mglT_2 + kl^2(T_2 - T_1) = 0, \tag{19}$$

where we have cancelled the exponential in each term. We can simplify a little writing

$$\frac{g}{l}T_1 + \frac{k}{m}(T_1 - T_2) = \omega^2 T_1, \tag{20}$$
$$\frac{g}{l}T_2 + \frac{k}{m}(T_2 - T_1) = \omega^2 T_2, \tag{21}$$

The easiest and cleanest way to solve this set of equation is to formulate them as a matrix eigenvalue problem $MT = \omega^2 IT$, where $I$ is the identity matrix and

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \quad M = \begin{bmatrix} \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} \end{bmatrix}. \tag{22}$$

This has a non-trivial solution when

$$\det(M - \omega^2 I) = \begin{vmatrix} \frac{g}{l} + \frac{k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} - \omega^2 \end{vmatrix} = 0, \tag{23}$$

which gives a second order equation for the eigenvalue $\omega^2$:

$$\omega^4 - 2\left(\frac{g}{l} + \frac{k}{m}\right)\omega^2 + \left(\frac{g}{l}\right)^2 + 2\frac{g}{l}\frac{k}{m} = 0, \tag{24}$$

with solutions

$$\omega_{1,2}^2 = \left(\frac{g}{l}+\frac{k}{m}\right) \pm \sqrt{\left(\frac{g}{l}+\frac{k}{m}\right)^2 - 2\frac{g}{l}\frac{k}{m}}$$
$$= \frac{g}{l}+\frac{k}{m} \pm \frac{k}{m}. \quad (25)$$

For the frequency $\omega_1 = \sqrt{\frac{g}{l}}$, inserted back into Eqs. (20) and (21) we get that $T_1 = T_2$ for the solution (eigenvector). The physical solution is a linear combination of the positive and negative angular frequency solutions,

$$\theta_1(t) = \frac{1}{2}T_1 e^{i\omega_1 t} + \frac{1}{2}T_1 e^{-i\omega_1 t} = T_1\cos(\omega_1 t), \quad (26)$$
$$\theta_2(t) = \frac{1}{2}T_2 e^{i\omega_1 t} + \frac{1}{2}T_2 e^{-i\omega_1 t} = T_1\cos(\omega_1 t), \quad (27)$$

so the two pendula swing in phase with angular frequency $\omega_1$. For the frequency $\omega_2 = \sqrt{\frac{g}{l}+\frac{2k}{m}}$ we get $T_1 = -T_2$. Here

$$\theta_1(t) = \frac{1}{2}T_1 e^{i\omega_2 t} + \frac{1}{2}T_1 e^{-i\omega_2 t} = T_1\cos(\omega_2 t), \quad (28)$$
$$\theta_2(t) = \frac{1}{2}T_2 e^{i\omega_2 t} + \frac{1}{2}T_2 e^{-i\omega_2 t} = -T_1\cos(\omega_2 t), \quad (29)$$

so the pendula swing in opposite directions, exactly out of phase, with angular frequency $\omega_2$.

## Question 2 Compton scattering redux

In Compton scattering a photon $\gamma$ with initial energy $E_\gamma$ scatters of a charged particle with mass $m$ at rest. The angle of scattering is $\theta$.

We remind you that the energy of a photon is given in terms of its frequency $\nu$ and wavelength $\lambda$ as $E = h\nu = \frac{hc}{\lambda}$, where $h$ is Planck's constant.

**a)** Draw a sketch of the process and give the equations for the conservation of relativistic energy and momentum in the collision in terms of the four-momenta of the particles $p_\gamma^\mu$ and $p_m^\mu$. [3 points]

**Answer:** The conservation of relativistic energy and momentum can be expressed in terms of the conservation of the total four-momenta of the two particles before and after the collision. If we use primes to denote the momenta after the collision we have

$$p_\gamma^\mu + p_m^\mu = p'^\mu_\gamma + p'^\mu_m. \quad (30)$$

A sketch of the process can be found in Fig. 2.

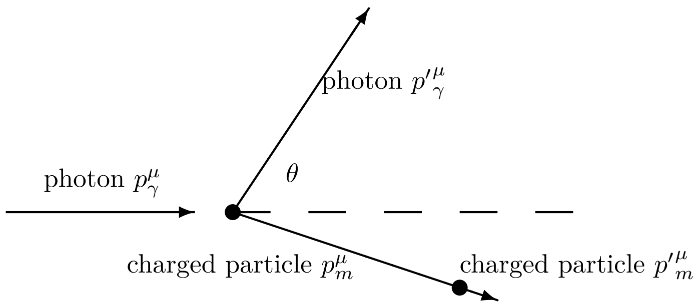


Figure 2: Compton scattering of a photon at an angle $\theta$.

**b)** Show that we can write

$$p_m(p_\gamma - p'_\gamma) = p_\gamma p'_\gamma, \tag{31}$$

where $p_a p_b = p^\mu_a p_{b\mu}$ means the contraction of the two four-vectors $p_a$ and $p_b$, and where the primes signify the four-momenta after the scattering. [4 points]

**Answer:** If we rearrange Eq. (30) such that $p'^{\mu}_m = p^\mu_\gamma + p^\mu_m - p'^{\mu}_\gamma$, we can square both sides of this expression such that

$$p'^2_m = p^2_\gamma + p^2_m + p'^2_\gamma + 2p_m(p_\gamma - p'_\gamma) - 2p_\gamma p'_\gamma. \tag{32}$$

Using that $p^2_\gamma = 0$ and $p^2_m = m^2c^2$ we have

$$m^2c^2 = m^2c^2 + 2p_m(p_\gamma - p'_\gamma) - 2p_\gamma p'_\gamma, \tag{33}$$

which gives $p_m(p_\gamma - p'_\gamma) = p_\gamma p'_\gamma$.

**c)** Use the above to derive Compton's formula

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta), \tag{34}$$

where $\lambda$ is the wavelength of the photon. *Nota bene!* We will not give points for other derivations of this expression. [6 points]

**Answer:** Since $E_\gamma = |\vec{p}_\gamma|c$ we have

$$p_\gamma p'_\gamma = \frac{E_\gamma}{c}\frac{E'_\gamma}{c} - \vec{p}_\gamma \cdot \vec{p}'_\gamma = \frac{E_\gamma E'_\gamma}{c^2}(1 - \cos\theta). \tag{35}$$

The charged particle is initially at rest so $p_m^\mu = (E/c, 0) = (mc, 0)$. This gives

$$p_m(p_\gamma - p'_\gamma) = mc\frac{E_\gamma}{c} - mc\frac{E'_\gamma}{c} = m(E_\gamma - E'_\gamma). \tag{36}$$

Using the result from **b)** we now have

$$E_\gamma - E'_\gamma = \frac{E_\gamma E'_\gamma}{mc^2}(1 - \cos\theta), \tag{37}$$

or

$$\frac{1}{E'_\gamma} - \frac{1}{E_\gamma} = \frac{1}{mc^2}(1 - \cos\theta). \tag{38}$$

The energy of a photon is related to its wavelength through $E_\gamma = h\nu = \frac{hc}{\lambda}$, so

$$\frac{\lambda'}{hc} - \frac{\lambda}{hc} = \frac{1}{mc^2}(1 - \cos\theta), \tag{39}$$

and finally

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta), \tag{40}$$

**d)** What is the energy of the outgoing photon for backward scattering, $\theta = \pi$, when the energy of the incoming photon is much larger than the rest energy of the charged particle? Are you surprised? [3 points]

**Answer:** The rest energy of the charged particle is $E_0 = mc^2$. Inserting $\theta = \pi$ in Eq. (38) gives

$$\frac{1}{E'_\gamma} - \frac{1}{E_\gamma} = \frac{2}{E_0}. \tag{41}$$

Since $E_\gamma \gg E_0$, $1/E_0 \gg 1/E_\gamma$ and so

$$\frac{1}{E'_\gamma} \simeq \frac{2}{E_0}, \tag{42}$$

or $E'_\gamma \simeq \frac{1}{2}E_0 = \frac{1}{2}mc^2$. Personally, I still find it a bit weird that the outgoing energy is dependent on the mass it hit.

**e)** Finally, let us look at so-called inverse Compton scattering where in the laboratory frame the charged particle is highly relativistic and makes a head-on collision with the photon, and where we assume that the energy of the photon in the rest frame of the charged particle is much less than the rest mass. Find the energy of a backscattered ($\theta = \pi$) photon as a function of its initial energy in the laboratory frame and the $\gamma$-factor for boosts between the charged particle rest frame and the laboratory frame. [5 points]

**Answer:** Because energy is in the first coordinate of the four-momentum, or rather $E/c$, it transforms as the first coordinate of the position four-vector, $ct$, so the Lorentz transformation of energy must be

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta p\right), \tag{43}$$

or, for a photon with $E_\gamma = cp_\gamma$ in all reference frames,

$$E'_\gamma = \gamma(1-\beta)E_\gamma, \tag{44}$$

where $E_\gamma$ is the energy before the boost and $E'_\gamma$ after. The direction of the velocity/boost is here in the direction of motion of the photon.

The energy of the photon in the rest frame of the charged particle is now given by $E'_\gamma = \gamma(1+\beta)E_\gamma$ because the boost must be in the direction of movement of the charged particle for the new reference frame to "catch up" with the particle. After the scattering in the rest frame of the particle Eq. (38) gives that the energy is approximately the same same since $E'_\gamma \ll E_0$, but the photon is now travelling in the opposite direction.

This energy must now be boosted back to the original frame. The boost is in the opposite direction, but since the photon has also changed direction, the energy of the scattered photon in the laboratory frame becomes

$$E''_\gamma \simeq \gamma(1+\beta)E'_\gamma = \gamma^2(1+\beta)^2E_\gamma \simeq 4\gamma^2E_\gamma, \tag{45}$$

where we have used that the charged particle is highly relativistic so that $\beta \simeq 1$.

**Question 3 Synchrotron radiation**

We will begin by looking at a single non-relativistic charged particle in circular motion. For concreteness, let the particle move in a circle with radius $R$ around the origin of the $(x,y)$-plane and with angular velocity $\omega$.

Let us remind you that the electric radiation field far away from a charge and current distribution can be written as

$$\vec{E}_{\text{rad}}(\vec{r},t) = \frac{\mu_0}{4\pi r}\left((\ddot{\vec{p}} \times \hat{n}) \times \hat{n} - \frac{1}{c}\ddot{\vec{m}} \times \hat{n}\right)_{\text{ret}} \tag{46}$$

where $\hat{n} = \vec{r}/r$ is a unit vector in the direction of the observer.

**a)** Find the electric dipole moment the particle. [3 points]

**Answer:** The charge density of a single charged particle is given by $\rho(\vec{r},t) = q\delta(\vec{r} - \vec{r}(t))$, where $\vec{r}(t)$ is the path of the particle and $q$ is the charge. The electric dipole moment is then

$$\vec{p} = \int \vec{r}\rho(\vec{r},t)dV = \int \vec{r}e\delta(\vec{r} - \vec{r}(t))dV = e\vec{r}(t), \tag{47}$$

where $\vec{r}(t)$ forms a circular path around the accelerator ring.

**b)** Find the magnetic dipole moment of the particle. [3 points]

**Answer:** The current density of a single charged particle is given by $\vec{j}(\vec{r}, t) = q\vec{v}(t)\delta(\vec{r} - \vec{r}(t))$, where $\vec{v}(t)$ is the velocity of the proton. The magnetic dipole moment is then

$$\vec{m} = \frac{1}{2}\int \vec{r} \times \vec{j}(\vec{r}, t)dV = \frac{1}{2}\int \vec{r} \times e\vec{v}(t)\delta(\vec{r} - \vec{r}(t))dV = \frac{e}{2}\vec{r}(t) \times \vec{v}(t). \tag{48}$$

Here the magnetic moment is in the direction of angular momentum, $\vec{\ell} = \vec{r} \times \vec{p}$, perpendicular to the circle.

**c)** Find expressions for the resulting radiation fields far away from the source and in same the plane as the circular motion. What is the polarisation and the wavelength of this radiation? [7 points]

**Answer:** We have $\ddot{\vec{p}} = q\vec{a}(t)$ and since the magnetic moment (just like the angular momentum) is constant for circular motion — the directions of $\vec{r}(t)$ and $\vec{v}(t)$ change, but not their magnitude, nor the direction of their cross product — we have $\ddot{\vec{m}} = 0$. Thus

$$\vec{E}_{\rm rad}(\vec{r}, t) = \frac{\mu_0}{4\pi r}((\ddot{\vec{p}} \times \hat{n}) \times \hat{n})_{\rm ret} = \frac{\mu_0 q}{4\pi r}(\vec{a}(t_r) \times \hat{n}) \times \hat{n}, \tag{49}$$

where $t_r = t - r/c$ is the retarded time.

Let us place our coordinate system such that the observer is on the $x$-axis. Then $\hat{n} = \hat{e}_x$. With the charged particle moving counter-clockwise we can parametrise the its path as $\vec{r}(t) = R(\cos\omega t\,\hat{e}_x + \sin\omega t\,\hat{e}_y)$. Then $\vec{a}(t) = -\omega^2 R(\cos\omega t\,\hat{e}_x + \sin\omega t\,\hat{e}_y)$. Input into the expression for the electric field we get

$$\begin{aligned}
\vec{E}_{\rm rad}(\vec{r}, t) &= -\omega^2 R\frac{\mu_0 q}{4\pi r}((\cos\omega t_r\,\hat{e}_x + \sin\omega t_r\,\hat{e}_y) \times \hat{e}_x) \times \hat{e}_x \\
&= -\omega^2 R\frac{\mu_0 q}{4\pi r}(-\sin\omega t_r\,\hat{e}_z) \times \hat{e}_x \\
&= \frac{\mu_0 q\omega^2 R}{4\pi r}\sin(\omega(t - r/c))\,\hat{e}_y.
\end{aligned} \tag{50}$$

The magnetic field is given by

$$\vec{B}_{\rm rad}(\vec{r}, t) = \frac{1}{c}\hat{n}\times\vec{E}_{\rm rad} = \frac{1}{c}\hat{e}_x\times\vec{E}_{\rm rad} = \frac{\mu_0 q\omega^2 R}{4\pi rc}\sin(\omega(t-r/c))\,\hat{e}_z. \tag{51}$$

Since the fields oscillate on a fixed axis, the radiation is linearly polarised. The frequency of the radiation is given by the angular frequency of the charged particle in the ring. This is $\omega = v/2\pi R$, so the wavelength is

$$\lambda = \frac{c}{\nu} = \frac{c}{2\pi\omega} = \frac{c}{2\pi v/2\pi R} = \frac{R}{\beta}. \tag{52}$$

The radiation from a particle undergoing acceleration perpendicular to its direction of motion, say in a circular accelerator, is called *synchrotron radiation.* Here we will investigate this radiation for the Large Hadron Collider (LHC) at CERN. The LHC accelerates protons of mass $m_p = 938.2\,\mathrm{MeV/c^2}$ to energies of $E_p = 6.5\,\mathrm{TeV/c}$ around a ring of radius $R = 2804\,\mathrm{m}$.[1]

**d)** What is the instantaneous inertial rest frame and why is Larmor's formula, as given in the formulae collection, valid there? [3 points]

**Answer:** The instantaneous inertial rest frame of a particle is the reference frame where the particle is at rest. If the particle changes velocity then this frame changes, so the frame moves with the particle's world line. Larmor's formula for the power radiated by a non-relativistic charged particle is valid in this frame since the particle is at rest, thus definitely non-relativistic. The particle is still accelerated in this frame since the acceleration in this frame, called the proper acceleration, can be non-zero.

**e)** Show that the relativistic form of Larmor's formula is

$$P = \frac{\mu_0 q^2}{6\pi c}\left[\gamma^4 a^2 + \gamma^6 \frac{(\vec{v}\cdot\vec{a})^2}{c^2}\right]. \tag{53}$$

*Hint:* You may assume that the radiated power from an accelerated charge is a Lorentz invariant quantity. This is proven in the lecture notes. [5 points]

**Answer:** The non-relativistic form of Larmor's formula is

$$P = \frac{\mu_0 q^2}{6\pi c} a^2. \tag{54}$$

Since this must apply in the instantaneous inertial rest frame where the acceleration is the proper acceleration, $a = a_0$, and since we know that the square of the proper acceleration is a Lorentz invariant because the contraction of the four-acceleration with itself yields $A^\mu A_\mu = -a_0^2$, we need to find an expression for $a_0^2$.

Since $U^\mu = \gamma(c, \vec{v})$ we have that

$$\begin{aligned} A^\mu &= \frac{dU^\mu}{d\tau} = \frac{d}{d\tau}\gamma(c,\vec{v}) \\ &= \frac{dt}{d\tau}\frac{d}{dt}\gamma(c,\vec{v}) = \gamma\frac{d}{dt}\gamma(c,\vec{v}) \\ &= \gamma\frac{d\gamma}{dt}(c,\vec{v}) + \gamma^2(0,\vec{a}), \end{aligned} \tag{55}$$

[1] 1 TeV = $10^{12}$ eV.

using $dt = \gamma d\tau$. We need the time derivative of the $\gamma$

$$\begin{aligned}\frac{d\gamma}{dt} &= \frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \\ &= -\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{1}{c^2}\right)\frac{d}{dt}v^2 \\ &= \gamma^3\frac{\vec{v}\cdot\vec{a}}{c^2},\end{aligned} \tag{56}$$

since $\frac{d}{dt}v^2 = \frac{d}{dt}\vec{v}\cdot\vec{v} = 2\vec{v}\cdot\frac{d}{dt}\vec{v} = 2\vec{v}\cdot\vec{a}$. Then

$$A^\mu = \gamma^4\frac{\vec{v}\cdot\vec{a}}{c^2}(c,\vec{v}) + \gamma^2(0,\vec{a}) = (\gamma^4\frac{\vec{v}\cdot\vec{a}}{c}, \gamma^4\frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} + \gamma^2\vec{a}), \tag{57}$$

and

$$\begin{aligned}A^\mu A_\mu &= \gamma^4\frac{\vec{v}\cdot\vec{a}}{c}\gamma^4\frac{\vec{v}\cdot\vec{a}}{c} - \left(\gamma^4\frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} + \gamma^2\vec{a}\right)^2 \\ &= \gamma^8\frac{(\vec{v}\cdot\vec{a})^2}{c^2} - \gamma^8\frac{(\vec{v}\cdot\vec{a})^2}{c^4}\vec{v}\cdot\vec{v} - 2\gamma^6\frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v}\cdot\vec{a} - \gamma^4\vec{a}\cdot\vec{a} \\ &= \gamma^8\frac{(\vec{v}\cdot\vec{a})^2}{c^2}(1-\beta^2) - 2\gamma^6\frac{(\vec{v}\cdot\vec{a})^2}{c^2} - \gamma^4 a^2 \\ &= -\gamma^6\frac{(\vec{v}\cdot\vec{a})^2}{c^2} - \gamma^4 a^2,\end{aligned} \tag{58}$$

where we have used that $\gamma^2 = 1/(1-\beta^2)$. Thus

$$P = \frac{\mu_0 q^2}{6\pi c}a_0^2 = \frac{\mu_0 q^2}{6\pi c}\left(\gamma^6\frac{(\vec{v}\cdot\vec{a})^2}{c^2} + \gamma^4 a^2\right). \tag{59}$$

**f)** Show that the radiated power in a circular accelerator such as the LHC can be expressed as

$$P \simeq \frac{cq^2}{6\pi\epsilon_0}\frac{\gamma^4}{R^2}. \tag{60}$$

and find the radiation energy loss for a proton at the LHC. For reference $\epsilon_0 = 8.85\cdot 10^{-12}\,\mathrm{C^2/Nm^2}$ and the charge of a proton is $q = 1.60\cdot 10^{-19}\,\mathrm{C}$. What happens if we instead try to use an electron with mass $m_e = 0.511\,\mathrm{MeV/c^2}$? [5 points]

**Answer:** Using Eq. (53) we have in the lab frame that $\vec{v}\cdot\vec{a} = 0$ and thus

$$P = \frac{\mu_0 q^2}{6\pi c}(\gamma^4 a^2) = \frac{\mu_0 q^2}{6\pi c}\gamma^4\left(\frac{v^2}{R}\right)^2 \simeq \frac{cq^2}{6\pi\epsilon_0}\frac{\gamma^4}{R^2} \tag{61}$$

since the particles must be highly relativistic and where we have used $c^2 = 1/\epsilon_0\mu_0$. For a proton in the LHC the $\gamma$-factor is

$$\gamma = \frac{E_p}{m_p c^2} = \frac{6.5\,\mathrm{TeV}}{938.2\,\mathrm{MeV}} \simeq 6930, \tag{62}$$

giving an energy loss of

$$P = \frac{(3.0 \cdot 10^8\,\mathrm{m/s}) \cdot (1.60 \cdot 10^{-19}\,\mathrm{C})^2}{6\pi \cdot 8.85 \cdot 10^{-12}\,\mathrm{C^2/Nm^2}} \frac{6930^4}{(2804\,\mathrm{m})^2} = 1.35 \cdot 10^{-11}\,\mathrm{J/s}. \tag{63}$$

This may seem small, but given that the LHC circulates 2808 bunches of $1.2 \cdot 10^{11}$ protons per bunch in two counter circulating beams, the total energy loss is $2 \cdot 2808 \cdot 1.2 \cdot 10^{11} \cdot 1.35 \cdot 10^{-11}\,\mathrm{J/s} = 9100\,\mathrm{J/s}$, or just above 9k watt.

The lighter electron has a much larger gamma factor $\gamma_e = (m_p/m_e)\gamma_p = 1.2 \cdot 10^7$. Since the power goes as $\gamma^4$ this results in an energy loss of around 150 watt *per electron.* This would be completely unfeasible to run with any significant number of electrons.

**g)** Explain why the fields you found in sub-question **c)** are not compatible with Eq. (53). What would be needed to find the radiation fields from the LHC? [3 points]

**Answer:** The problem with the fields in sub-question **c)** is that we have assume that the particle moves non-relativistically. We can find the energy flux with Poynting's vector which is given as $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$. This flux does not have any of the necessary $\gamma$-factors for the integral over a surface around the LHC to yield the relativistic Larmor formula. In calculating the electric (and magnetic) dipole moments we have performed integrals over the source coordinates, these would be different in the lab frame of the observer. In order to find the fields in the relativistic case we should start from the general expression of the Lienard-Wiechert potentials.