# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

**Exam in: FYS3120 Classical Mechanics and Electrodynamics**
**Day of exam: 11 June 2018**
**Exam hours: 14:30 (4 hours)**
**This examination paper consists of 5 (five) pages.**
**Permitted materials:** Approved calculator
Øgrim and Lian: "Fysiske størrelser og enheter"
Angell and Lian: "Fysiske størrelser og enheter"
Rottman: "Matematisk formelsamling".
Collection of formulae for FYS3120

*Make sure that your copy of this examination paper is complete before answering.*

*You may answer the questions either in English or Norwegian.*

**Question 1 Swinging Lagrangian mechanics**

A mechanical system consists of two pendula with mass $m$ and length $l$ swinging from fixed points a distance $d$ apart. The pendula are coupled by an (effectively weightless) spring with spring constant $k$, and move under the influence of gravity. The pendula swing in the same plane and the spring has an unstretched length $d_0$, *viz.* the length when no spring force is acting. See illustration in Fig. 1.

We remind you that the potential energy for a spring is given by $V = \frac{1}{2}kx^2$, when $x$ is the displacement of the string length.

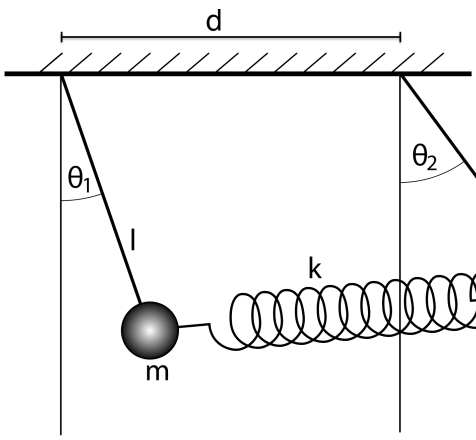


Figure 1: Two coupled pendula.

**a)** How many degrees of freedom does this system have? Explicitly give your choice of generalised coordinates. [3 points]

**b)** Find the potential energy of the system in terms of the generalised coordinates assuming that the angles $\theta_1$ and $\theta_2$ are small. We shall keep to this assumption in the following. *Hint:* the small angle expansions of sine and cosine to second order in angles are

$$\sin\theta = \theta + \mathcal{O}(\theta^3), \quad \cos\theta = 1 - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^4). \tag{1}$$

[5 points]

**c)** Find the equilibrium position of the pendula. [4 points]

**d)** Show that the Lagrangian of the system can be written

$$L = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) - \frac{1}{2}k(d - d_0 + l(\theta_2 - \theta_1))^2, \tag{2}$$

where $g$ is the acceleration due to gravity. [3 points]

**e)** Find the equations of motion. [4 points]

**f)** If $d = d_0$, show that

$$\theta_i(t) = T_i e^{\pm i\omega t}, \tag{3}$$

are solutions and find the *two* different allowed *magnitudes* of the angular frequency $\omega$. Briefly discuss the physical interpretation of these solutions. [6 points]

## Question 2 Compton scattering redux

In Compton scattering a photon $\gamma$ with initial energy $E_\gamma$ scatters of a charged particle with mass $m$ at rest. The angle of scattering is $\theta$.

We remind you that the energy of a photon is given in terms of its frequency $\nu$ and wavelength $\lambda$ as $E = h\nu = \frac{hc}{\lambda}$, where $h$ is Planck's constant.

**a)** Draw a sketch of the process and give the equations for the conservation of relativistic energy and momentum in the collision in terms of the four-momenta of the particles $p_\gamma^\mu$ and $p_m^\mu$. [3 points]

**b)** Show that we can write

$$p_m(p_\gamma - p'_\gamma) = p_\gamma p'_\gamma, \tag{4}$$

where $p_a p_b = p_a^\mu p_{b\mu}$ means the contraction of the two four-vectors $p_a$ and $p_b$, and where the primes signify the four-momenta after the scattering. [4 points]

**c)** Use the above to derive Compton's formula

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta), \tag{5}$$

where $\lambda$ is the wavelength of the photon. *Nota bene!* We will not give points for other derivations of this expression. [6 points]

**d)** What is the energy of the outgoing photon for backward scattering, $\theta = \pi$, when the energy of the incoming photon is much larger than the rest energy of the charged particle? Are you surprised? [3 points]

**e)** Finally, let us look at so-called inverse Compton scattering where in the laboratory frame the charged particle is highly relativistic and makes a head-on collision with the photon, and where we assume that the energy of the photon in the rest frame of the charged particle is much less than the rest mass. Find the energy of a backscattered ($\theta = \pi$) photon as a function of its initial energy in the laboratory frame and the $\gamma$-factor for boosts between the charged particle rest frame and the laboratory frame. [5 points]

**Question 3 Synchrotron radiation**

We will begin by looking at a single non-relativistic charged particle in circular motion. For concreteness, let the particle move in a circle with radius $R$ around the origin of the $(x, y)$-plane and with angular velocity $\omega$.

Let us remind you that the electric radiation field far away from a charge and current distribution can be written as

$$\vec{E}_{\mathrm{rad}}(\vec{r}, t) = \frac{\mu_0}{4\pi r}\left( (\ddot{\vec{p}} \times \hat{n}) \times \hat{n} - \frac{1}{c}\ddot{\vec{m}} \times \hat{n} \right)_{\mathrm{ret}}, \tag{6}$$

where $\hat{n} = \vec{r}/r$ is a unit vector in the direction of the observer.

**a)** Find the electric dipole moment the particle. [3 points]

**b)** Find the magnetic dipole moment of the particle. [3 points]

**c)** Find expressions for the resulting radiation fields far away from the source and in same the plane as the circular motion. What is the polarisation and the wavelength of this radiation? [7 points]

The radiation from a particle undergoing acceleration perpendicular to its direction of motion, say in a circular accelerator, is called *synchrotron radiation*. Here we will investigate this radiation for the Large Hadron Collider (LHC) at CERN. The LHC accelerates protons of mass $m_p = 938.2\,\mathrm{MeV/c^2}$ to energies of $E_p = 6.5\,\mathrm{TeV/c}$ around a ring of radius $R = 2804\,\mathrm{m}$.[1]

**d)** What is the instantaneous inertial rest frame and why is Larmor's formula, as given in the formulae collection, valid there? [3 points]

**e)** Show that the relativistic form of Larmor's formula is

$$P = \frac{\mu_0 q^2}{6\pi c}\left[\gamma^4 a^2 + \gamma^6 \frac{(\vec{v} \cdot \vec{a})^2}{c^2}\right]. \tag{7}$$

*Hint:* You may assume that the radiated power from an accelerated charge is a Lorentz invariant quantity. This is proven in the lecture notes. [5 points]

**f)** Show that the radiated power in a circular accelerator such as the LHC can be expressed as

$$P \simeq \frac{cq^2}{6\pi\epsilon_0}\frac{\gamma^4}{R^2}. \tag{8}$$

and find the radiation energy loss for a proton at the LHC. For reference $\epsilon_0 = 8.85 \cdot 10^{-12}\,\mathrm{C^2/Nm^2}$ and the charge of a proton is $q = 1.60 \cdot 10^{-19}\,\mathrm{C}$. What happens if we instead try to use an electron with mass $m_e = 0.511\,\mathrm{MeV/c^2}$? [5 points]

[1] $1\ \mathrm{TeV} = 10^{12}\ \mathrm{eV}$.

**g)** Explain why the fields you found in sub-question **c)** are not compatible with Eq. (7). What would be needed to find the radiation fields from the LHC? [3 points]