

# FYS 3120: Classical Mechanics and Electrodynamics

## Formula Collection

### 1 Analytical Mechanics

#### The Lagrangian

$$L = L(q, \dot{q}, t), \quad (1)$$

is a function of the *generalized coordinates*  $q = \{q_i; i = 1, 2, \dots, d\}$  of the physical system, and their time derivatives  $\dot{q} = \{\dot{q}_i; i = 1, 2, \dots, d\}$ . The Lagrangian may also have an *explicit* dependence of time  $t$ .

#### Lagrange's equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, d. \quad (2)$$

There is one equation for each generalized coordinate.

#### Generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, 2, \dots, d. \quad (3)$$

is also referred to as *canonical* or *conjugate* momentum. There is one generalized momentum  $p_i$  conjugate to each generalized coordinate  $q_i$ .

#### The Hamiltonian

$$H(p, q) = \sum_{i=1}^d \dot{q}_i p_i - L \quad (4)$$

is usually considered as a function of the generalized coordinates  $q_i$  and momenta  $p_i$ .

#### Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \dots, d \quad (5)$$

(6)

#### Standard expressions for $L$ og $H$

$$\begin{aligned} L &= K - V \\ H &= K + V \end{aligned} \quad (7)$$

with  $K$  as kinetic energy and  $V$  as potential energy. There are cases where  $H$  is *not* the total energy.

#### Charged particle in electromagnetic field (non-relativistic)

$$\begin{aligned} L = L(\mathbf{r}, \mathbf{v}) &= \frac{1}{2} m v^2 - e\phi + e\mathbf{v} \cdot \mathbf{A} \\ H = H(\mathbf{r}, \mathbf{p}) &= \frac{1}{2m} (\mathbf{p} - e\mathbf{A})^2 + e\phi \end{aligned} \quad (8)$$

## 2 Relativity

### Space-time coordinates

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{r}) \quad (9)$$

### General Lorentz transformation

$$x^\mu \rightarrow x'^\mu = L^\mu_\nu x^\nu + a^\mu \quad (10)$$

### Special Lorentz transformation with velocity $v$ in the $x$ direction

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \end{aligned} \quad (11)$$

with  $\beta = v/c$  and  $\gamma = 1/\sqrt{1 - \beta^2}$ , and  $x^2$  og  $x^3$  are unchanged.

### Condition satisfied by Lorentz transformation matrices

$$g_{\mu\nu} L^\mu_\rho L^\nu_\sigma = g_{\rho\sigma} \quad (12)$$

### Invariant line element

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \mathbf{r}^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu = \Delta x_\mu \Delta x^\mu \quad (13)$$

### Metric tensor

$$g_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu = 0 \\ -1, & \mu = \nu \neq 0 \end{cases}$$

### Upper and lower index

$$\begin{aligned} x_\mu &= g_{\mu\nu} x^\nu, & (x^\mu) &= (ct, \mathbf{r}), & (x_\mu) &= (ct, -\mathbf{r}) \\ x^\mu &= g^{\mu\nu} x_\nu, & g_{\mu\rho} g^{\rho\nu} &= \delta_\mu^\nu \end{aligned} \quad (14)$$

### Proper time - time dilatation

$$d\tau = \frac{1}{c} \sqrt{ds^2} = \frac{1}{\gamma} dt, \quad (15)$$

$d\tau$ : proper time interval = time measured in an (instantaneous) rest frame of a moving body (by a co-moving clock)

$ds^2$ : invariant line element of an infinitesimal section of the object's world line

$dt$ : coordinate time interval = time interval measured in arbitrarily chosen inertial system

### Length contraction

$$L = \frac{1}{\gamma} L_0 \quad (16)$$

Lengths of a moving body measured in the direction of motion.

$L_0$ : length measured in the rest frame of a moving body

$L$ : length measured (at simultaneity) in an arbitrarily chosen inertial frame.

### Four velocity

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \mathbf{v}), \quad U^\mu U_\mu = c^2 \quad (17)$$

### Four acceleration

$$\mathcal{A}^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}, \quad \mathcal{A}^\mu U_\mu = 0 \quad (18)$$

### Proper acceleration $\mathbf{a}_0$

Acceleration measured in instantaneous rest frame,

$$\mathcal{A}^\mu \mathcal{A}_\mu = -\mathbf{a}_0^2 \quad (19)$$

### Four momentum

$$p^\mu = m U^\mu = m\gamma(c, \mathbf{v}) = \left(\frac{E}{c}, \mathbf{p}\right) \quad (20)$$

with  $m$  as the (rest) mass of a moving body.

### Relativistic energy

$$E = \gamma m c^2 \quad (21)$$

$\gamma m$  is sometimes referred to as the *relativistic mass* of the moving body.

### 3 Electrodynamics

#### Maxwell's equations

$$\begin{aligned}
 \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
 \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E} &= \mu_0 \mathbf{j} \\
 \nabla \cdot \mathbf{B} &= 0 \\
 \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} &= 0
 \end{aligned} \tag{22}$$

#### Maxwell's equations in covariant form

$$\begin{aligned}
 \partial_\mu F^{\mu\nu} &= \mu_0 j^\nu, \quad \partial_\nu \equiv \frac{\partial}{\partial x^\nu} \\
 \partial_\mu \tilde{F}^{\mu\nu} &= 0, \quad \tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}
 \end{aligned} \tag{23}$$

#### Electromagnetic field tensor

$$\begin{aligned}
 F^{k0} &= \frac{1}{c} E_k, \quad F^{ij} = -\epsilon_{ijk} B_k \\
 \tilde{F}^{k0} &= B_k, \quad \tilde{F}^{ij} = \frac{1}{c} \epsilon_{ijk} E_k
 \end{aligned} \tag{24}$$

#### Four-current density

$$(j^\mu) = (c\rho, \mathbf{j}) \tag{25}$$

#### Charge conservation

$$\partial_\mu j^\mu = 0, \quad \frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j} = 0 \tag{26}$$

#### Electromagnetic potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t} \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{27}$$

#### Four potential

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (A^\mu) = \left(\frac{1}{c}\phi, \mathbf{A}\right) \tag{28}$$

#### Lorentz force

Force from the electromagnetic field on a point particle with charge  $q$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{29}$$

#### Potentials from charge and current distributions

in Lorentz gauge,  $\partial_\mu A^\mu = 0$ :

$$\begin{aligned}
 \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV' \\
 \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV'
 \end{aligned} \tag{30}$$

### Retarded time

$$t' = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'| \quad (31)$$

### Electric dipole moment

$$\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r})dV \quad (32)$$

### Electric dipole potential (dipole at the origin)

$$\phi = \frac{\mathbf{n} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2}, \quad \mathbf{n} = \frac{\mathbf{r}}{r} \quad (33)$$

### Force and torque (about the origin)

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}, \quad \mathbf{M} = \mathbf{p} \times \mathbf{E} \quad (34)$$

### Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{j}(\mathbf{r}) dV \quad (35)$$

### Magnetic dipole potential (dipole at the origin)

$$\mathbf{A} = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \mathbf{n}, \quad \mathbf{n} = \frac{\mathbf{r}}{r} \quad (36)$$

### Force and torque (about the origin)

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \text{ (current loop)}, \quad \mathbf{M} = \mathbf{m} \times \mathbf{B} \quad (37)$$

### Lorentz transformation of the electromagnetic field

$$F'^{\mu\nu} = L^\mu_\rho L^\nu_\sigma F^{\rho\sigma} \quad (38)$$

### Lorentz invariants

$$\begin{aligned} \mathbf{E}^2 - c^2\mathbf{B}^2 &= -\frac{c^2}{2}F_{\mu\nu}F^{\mu\nu} \\ \mathbf{E} \cdot \mathbf{B} &= \frac{c}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} \end{aligned} \quad (39)$$

### Special Lorentz transformations

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}) \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel}, \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}/c^2) \end{aligned} \quad (40)$$

The fields are decomposed in a parallel component ( $\parallel$ ), in the direction of transformation velocity  $\mathbf{v}$ , and a perpendicular component ( $\perp$ ), orthogonal to  $\mathbf{v}$ .

### Electromagnetic field energy density

$$u = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) = \frac{\epsilon_0}{2}(E^2 + c^2 B^2) \quad (41)$$

### Electromagnetic energy current density (Poynting's vector)

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (42)$$

### Monochromatic plane waves, plane polarized

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \quad \mathbf{E}_0 = E_0 \mathbf{e}_1 \\ \mathbf{B}(\mathbf{r}, t) &= \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t); \quad \mathbf{B}_0 = B_0 \mathbf{e}_2 \\ \mathbf{E}_0 \cdot \mathbf{k} &= \mathbf{B}_0 \cdot \mathbf{k} = 0, \quad \mathbf{B}_0 = \frac{1}{c} \mathbf{n} \times \mathbf{E}_0, \quad \mathbf{n} = \frac{\mathbf{k}}{k} \end{aligned} \quad (43)$$

### Monochromatic plane waves, circular polarized

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \text{Re}(\mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]) , \quad \mathbf{E}_0 = E_0 \frac{1}{\sqrt{2}} (\mathbf{e}_1 \pm i \mathbf{e}_2) \\ \mathbf{B}(\mathbf{r}, t) &= \text{Re}(\mathbf{B}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]) , \quad \mathbf{B}_0 = B_0 \frac{1}{\sqrt{2}} (\mathbf{e}_2 \mp i \mathbf{e}_1) \end{aligned} \quad (44)$$

### Polarization vectors

$$\mathbf{e}_1 \cdot \mathbf{k} = \mathbf{e}_2 \cdot \mathbf{k} = 0, \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1 \quad (45)$$

### Four-wave vector

$$(k^\mu) = \left( \frac{\omega}{c}, \mathbf{k} \right), \quad \omega = ck \quad (46)$$

### Radiation fields, in the wave zone ( $r \gg r', \lambda$ )

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) &= -\frac{\mu_0}{4\pi c r} \mathbf{n} \times \frac{d}{dt} \int \mathbf{j}(\mathbf{r}', t') dV', \quad \mathbf{n} = \frac{\mathbf{r}}{r} \\ \mathbf{E}(\mathbf{r}, t) &= c \mathbf{B}(\mathbf{r}, t) \times \mathbf{n} \end{aligned} \quad (47)$$

### Electric dipole radiation

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c r} \mathbf{n} \times \ddot{\mathbf{p}}(t - r/c), \quad \mathbf{E}(\mathbf{r}, t) = c \mathbf{B}(\mathbf{r}, t) \times \mathbf{n} \quad (48)$$

### Radiation from accelerated, charged particle

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi c r} [\mathbf{a} \times \mathbf{n}]_{ret}, \quad \mathbf{E}(\mathbf{r}, t) = c \mathbf{B}(\mathbf{r}, t) \times \mathbf{n}_{ret} \\ \mathbf{n} &= \mathbf{R}/R, \quad \mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t) \end{aligned} \quad (49)$$

with  $\mathbf{r}(t)$  as the particle's position vector.

### Radiated power, Larmor's formula

$$P = \frac{\mu_0 q^2}{6\pi c} \mathbf{a}^2 \quad (50)$$