FYS3140 Exam V2011. Short solution.

Problem 1

The integral does not depend on the sign of a, and we assume $a > 0$ and replace a by $|a|$ in the answer. We also notice that the integral changes sign if m changes sign. Thus, we assume $m > 0$.

We consider the complex integral $\oint_C \frac{ze^{imz}}{z^2 + a^2} dz$ where the path of integration C is along the real axis from –R to R, and along a semicircle Γ in the upper half plane. Then according to Jordans lemma the integral on Γ tend to zero as $R \to \infty$, and we have

$$\int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2 + a^2} dx = 2\pi i \operatorname{Re} s(ia) = \pi i e^{-ma}$$, and the general result for $m \neq 0$ and $a \neq 0$ is

$$\int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2} \frac{m}{|m|} e^{-|ma|} \quad .$$

The limit as $a \to 0$ is clearly $\frac{\pi}{2} \frac{m}{|m|}$, in accordance with the known integral $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

The result obtained above is obviously wrong for $m = 0$. In this case the integral on Γ does not go to zero as $R \to 0$, but takes the value $2\pi i \operatorname{Re} s(ia) = \pi i$ for $R > a$ and zero for $R < a$, resulting in the correct value zero for the integral along the real axis.

Problem 2

a) Solutions of the form $e^{\lambda x}$ with $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$, and the general solution

$$y(x) = C_1 e^x \cos x + C_2 e^x \sin x \, .$$

b) Variation of the constants gives the particular solution $y_p(x) = xe^x$, and consequently the general solution

$$y(x) = C_1 e^x \cos x + C_2 e^x \sin x + xe^x \, .$$

Problem 3

a) The Laplace transform $U(s, x)$ of the solution $u(x,t)$ is obtained from the differential equation $\frac{d}{dx} U(s,x) + 2xsU(s,x) = \frac{2x}{s^2} + 2x$. The solution of the homogeneous part of this equation is

$U_h(s,x) = C(s)e^{-sx^2}$, and a particular solution is $U_p(s,x) = \frac{1}{s^3} + \frac{1}{s}$.

The condition $u(0,t) = 1$ yields $C(s) = -\frac{1}{s^3}$, and the final solution is

$$u(x,t) = -\frac{1}{2}H(t-x^2)(t-x^2)^2 + \frac{1}{2}t^2 + 1 \ .$$

Problem 4

a) We notice that the function $f(x)$ is even, i.e. $b_n = 0$.

$a_n = \frac{2}{L}\int_0^L (L-x)\cos\frac{n\pi x}{L}dx = \frac{4L}{\pi^2 n^2}$ for odd n , $a_n = 0$ for even n, $n \geq 2$,

$a_0 = L$.

$$f(x) = \frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1,3,5,\cdots}\frac{1}{n^2}\cos\frac{n\pi x}{L} \ .$$

$f(x) = L$ for $x = 0$ gives

$$\sum_{n=1,3,5,\cdots}\frac{1}{n^2} = \frac{\pi^2}{8} \ .$$

b) The differential equation $y'(x) + y(x) = f(x)$.

Inserts the series for $f(x)$ from a) above. Seeks a Fourier series for y(x):

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \ .$$

Inserted in the diff. equation (term by term differentiation of the series), series on the left hand side and right hand side equal term by term (orthogonality) for both the cosine and sine series yields:

$a_0 = L$, $-\frac{\pi}{L}na_n + b_n = 0$ all $n > 0$, $\frac{\pi}{L}nb_n + a_n = \frac{4L}{\pi^2}\frac{1}{n^2}$ for $n = 1,3,5,\cdots$ gives

$$a_n = \frac{4L}{\pi^2 n^2}\frac{1}{\left(\frac{\pi n}{L}\right)^2 + 1} \ , \qquad b_n = \frac{4}{\pi n}\frac{1}{\left(\frac{\pi n}{L}\right)^2 + 1} \ , \quad n = 1,3,5,\cdots$$

$a_n = b_n = 0$ even n, $n > 0$.