Home exam FYS3140 V2011. Short solution

Problem 1

Differential equation: $x(x-1)y''+(3x-1)y'+y=0$.

a) Series solution: $y=\sum_{n=0}^{\infty} a_n x^{n+s}, \;\; a_0 \neq 0.$

$$\sum_{n=0} a_n \left[(n+s)(n+s-1)+3(n+s)+1\right]x^{n+s} - \sum_{n=-1} a_{n+1}\left[(n+1+s)(n+s)+(n+1+s)\right]x^{n+s} = 0.$$

$n=-1: \;\; a_0 s^2 = 0, \text{ and } s_1 = s_2 = 0.$

$$n \geq 0: \;\; a_{n+1} = a_n \frac{(n+s)(n+s+2)+1}{(n+s)(n+s+2)+1} = a_n.$$

$y(x) = a_0 \sum_{n=0}^{\infty} x^n = \frac{a_0}{1-x}$, convergent for |x|<1, but the solution $y_1(x)=\frac{1}{1-x}$ is ok for all $x \neq 1$.

b) $y_2(x)=C(x)y_1(x)$. Inserted in the differential equation: $C(x)=\ln x = \ln|x| + \text{constant}$

$y_2(x)=\frac{\ln|x|}{1-x}$. General solution: $y(x)=c_1 y_1(x)+c_2 y_2(x) = \frac{c_1}{1-x}+c_2\frac{\ln|x|}{1-x}$.

Regions of validity: x<0, 0<x<1 and x>1. Separate boundary conditions are required for these three regions to obtain unique solutions, i.e. to determine a set of c_1 and c_2 that applies for the relevant region.

Problem 2

Differential equation: $y''(t)-a^2y(t)=f(t)$. Initial conditions: $y(0)=y'(0)=0, \;\; t \geq 0$.

Note that the diff. equation and hence the solution is independent of the sign of a. Thus, in the following a actually means $|a|$.

$$y(t)=\int_0^{\infty} G(t,t')f(t')dt' \quad , \quad y'(t)=\int_0^{\infty}\frac{d}{dt}\left[G(t,t')\right]f(t')dt' = \int_0^{\infty} G'(t,t')f(t')dt' .$$

$y(0)=y'(0)=0$ ok if $G(0,t')=G'(0,t')=0.$

$G(t,t')=G_I(t,t')=A(t')e^{at}+B(t')e^{-at}, \;\; t<t',$

$G(t,t')=G_{II}(t,t')=C(t')e^{at}+D(t')e^{-at}, \;\; t>t'.$

The initial conditions yield $A(t')=B(t')=0, \text{ and } G_I(t,t')=0.$

Continuity at t=t': $C(t')e^{at'}+D(t')e^{-at'}=0.$

Derivative at t=t': $aC(t')e^{at'}-aD(t')e^{-at'}=1$.

$$G_{II}(t,t')=\frac{1}{2a}\left(e^{-at'}e^{at}-e^{at'}e^{-at}\right)=\frac{1}{a}\sinh a(t-t').$$

$$y(t)=\int_0^t G_{II}(t,t')f(t')dt'=\frac{1}{a}\int_0^t \sinh a(t-t')f(t')dt'.$$

b) $f(t)=k\delta(t-t_0),\ t_0>0$. $y(t)=0$ for $t<t_0$, $y(t)=\frac{k}{a}\sinh a(t-t_0)$ for $t>t_0$.

c) $y(t)=0$ for $t<t_1$, $y(t)=\frac{K}{a}\int_{t_1}^{t_2}\sinh a(t-t')dt'=-\frac{K}{a^2}\left[\cosh a(t-t_2)-\cosh a(t-t_1)\right]$, $t>t_2$

$$y(t)=\frac{K}{a}\int_{t_1}^{t}\sinh a(t-t')dt'=\frac{K}{a^2}\cosh a(t-t_1)-\frac{K}{a^2} \quad \text{for } t_1<t<t_2.$$

Problem 3

a) Fourier transform of $e^{-a|t|}$:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{at}e^{-i\omega t}dt+\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-at}e^{-i\omega t}dt=\frac{1}{\sqrt{2\pi}}\frac{2a}{\omega^2+a^2}.$$

Fourier transform of the diff. equation:

$$-\omega^2 Y(\omega)-k^2Y(\omega)=\frac{1}{\sqrt{2\pi}}\frac{2a}{\omega^2+a^2},$$

$$Y(\omega)=-\frac{2a}{\sqrt{2\pi}}\frac{1}{(\omega^2+a^2)(\omega^2+k^2)}.$$

b) $$y(t)=-\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega^2+a^2)(\omega^2+k^2)}d\omega.$$

The complex integral is worked out by use of Jordans lemma. First t>0 (upper half circle):

$$\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega^2+a^2)(\omega^2+k^2)}d\omega=2\pi i\left[\operatorname{Res}(ia)+\operatorname{Res}(ik)\right]=\frac{\pi}{k^2-a^2}\left[\frac{1}{a}e^{-at}-\frac{1}{k}e^{-kt}\right].$$

$$y(t)=\frac{a}{k^2-a^2}\left[\frac{1}{k}e^{-kt}-\frac{1}{a}e^{-at}\right] \quad t>0.$$

Then for t<0, lower half circle, negative direction of integration. Residues at $-ia$ and $-ik$.

$$y(t)=\frac{a}{k^2-a^2}\left[\frac{e^{kt}}{k}-\frac{e^{at}}{a}\right] \quad t<0.$$

Now, we notice that the solution should be independent of the sign of k. $k>0$ has been assumed so far. The complete solution for all t and k is then:

$$y(t)=\frac{a}{k^2-a^2}\left[\frac{e^{-|kt|}}{|k|}-\frac{e^{-a|t|}}{a}\right].$$

As expected we see that $y(t)\to 0$ for $t\to\pm\infty$, in agreement with the requirement for a valid Fourier transform of the diff.equation. The solution is seen to be continuous at t=0.

c) In the case $k^2 = a^2$ there are double poles at ia and $-ia$. The residues at the poles are:

$$\operatorname{Res}(ia) = -\frac{e^{-at}}{4a^3} i(at+1) \text{ for } t>0, \quad \operatorname{Res}(-ia) = -\frac{e^{at}}{4a^3} i(at-1) \text{ for } t<0.$$

$$y(t) = -\frac{1}{2a^2}\left[a\,|t|+1\right]e^{-a|t|}.$$

Problem 4

a) $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{\sinh\pi}{\pi}(-1)^n \frac{1+in}{1+n^2}.$

b) x=0: $S = \sum_{n=1}^{\infty}(-1)^n \frac{1}{1+n^2} = \frac{1}{2}\left[\frac{\pi}{\sinh\pi} - 1\right].$

c) $e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{1+n^2} - \sum_{n=1}^{\infty}(-1)^n \frac{n}{1+n^2}\sin nx\right].$

x=0 gives the same sum as in b).