Orthogonal sets of functions

Recall our standard homogeneous differential equation:

$$y''+P(x)y'+Q(x)y=0 .$$

Now, there is a special class of such equations that take the form

$$\frac{d}{dx}[p(x)y']+[q(x)+\lambda r(x)]y=0$$, with $r(x)>0$. Thus,

$$p(x)y''+p'(x)y'+[q(x)+\lambda r(x)]y=0.$$

Examples:

Legendre: $(1-x^2)y''-2xy'+n(n+1)y=0$, $(\lambda=n(n+1),\ \ p(x)=1-x^2,\ \ q(x)=0$, and $r(x)=1$).

Fourier: $y''+\left(\frac{n\pi}{L}\right)^2 y=0$, $p(x)=r(x)=1$, $q(x)=0$.

Hermite: $y''-2xy'+2ny=0$, multiplication by e^{-x^2} gives it the right form:

$$e^{-x^2}y''-2xe^{-x^2}y'+2ne^{-x^2}y=0,$$ with $p(x)=r(x)=e^{-x^2}$, and $q(x)=0$, $\lambda=2n$.

Laguerre: $xy''+(1-x)y'+\lambda y=0$, multiplied by e^{-x} to get the correct form:

$$xe^{-x}y''+(1-x)e^{-x}y'+\lambda e^{-x}y=0,$$ with $p(x)=xe^{-x}$, $r(x)=e^{-x}$, and $q(x)=0$.

We are actually investigating so called eigenvalue equations of the general form:

$$Dy+\lambda r(x)y=0$$, where the differential operator D has the form

$$D=p(x)\frac{d^2}{dx^2}+p'(x)\frac{d}{dx}+q(x) .$$

Boundary conditions are applied at $x=a$ and $x=b$, i.e. $x\in[a,b]$.

Legendre: $x\in[-1,1]$,

Fourier: $x\in[-L,L]$,

Hermite: $x\in<-\infty,\infty>$,

Laguerre: $x \in [0, \infty>$.

Notice that for all the equations listed above we have $p(a) = p(b)$ (at the boundaries).

The boundary conditions at $x = a$ and $x = b$ typically lead to a series of discrete eigenvalues λ_n , and corresponding solutions (eigenfunctions) $y_n(x)$.

Without proof (which may be rather demanding!) we state that the eigenfunctions are orthogonal:

$$\int_a^b r(x) y_n(x) y_m(x)^* \, dx = 0 \quad \text{for} \quad \lambda_n \neq \lambda_m .$$

Complete set of eigenfunctions:
If any function $f(x)$ (without infinite discontinuities) can be expanded in a convergent series in terms of the set of eigenfunctions $\{y_n(x)\}$ for $x \in [a,b]$ (or the appropriate open interval):

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x),$$

then we say that the set $\{y_n(x)\}$ is complete.

The coefficients a_n are determined by utilizing the orthogonality of the eigenfunctions:

$$\int_a^b f(x) r(x) y_m(x)^* \, dx = \sum_n a_n \int_a^b r(x) y_n(x) y_m(x)^* \, dx = a_m \int_a^b r(x) y_m(x) y_m(x)^* \, dx .$$

It is usual to normalize the eigenfunctions, i.e. multiply them by a constant $C(n)$ ($y_n(x)$ is replaced by $C(n) y_n(x)$), so that

$$\int_a^b r(x) y_m(x) y_m(x)^* \, dx = 1 . \quad \text{Thus,}$$

$$a_m = \int_a^b f(x) r(x) y_m(x)^* \, dx .$$

Hermitian operators
For which type of eigenvalue equations and boundary conditions do the eigenfunctions form a complete orthogonal set?
If the differential operator D with given boundary conditions at $x = a$ and $x = b$ obeys the condition

$$\int_a^b y_n(x)^* D y_m(x) dx = \int_a^b y_m(x) D y_n(x)^* dx \ ,$$

then D is a Hermitian operator.

For Hermitian operators it follows (without proof here):

1. The eigenvalues are real (cf. observable quantities in quantum mechanics)
2. The eigenfunctions are orthogonal (over $x \in [a,b]$)
3. The eigenfunctions form a complete set for $x \in [a,b]$.

The eigenvalue equations listed as examples above, are all of the Hermitian type.

Examples:

Legendre equation: $(1-x^2)y_n'' - 2xy_n' + n(n+1)y_n = 0.$

Solutions: Legendre polynomials $y_n(x) = P_n(x)$ for $x \in [-1,1]$

Orthogonality: $\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1}\delta_{nm}$.

Completeness: $f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \ \ x \in [-1,1]. \quad a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x) dx$.

Hermite equation: $y_n'' - 2xy_n' + 2ny_n = 0$, $r(x) = e^{-x^2}$.

Solutions: Hermite polynomials: $y_n(x) = H_n(x)$ for $x \in <-\infty, \infty>$.

Orthogonality: $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi}\delta_{nm}.$

Completeness: $f(x) = \sum_{n=0}^{\infty} a_n H_n(x), \ \ \mathrm{a}_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x) dx.$