Week 4 – Plan

Lectures:

- Highlights of last week
- Laurent series
- Poles and zeros
- Start Residue theory

Problem session: work on problem set 3. Hand in by Monday 15th at 14:30. NO LECTURE THURSDAY.

Week 4 – Highlights of last week (1)

Integrals of complex functions: Independence of path

- If f is continuous in a domain D and has an antiderivative, then contour integrals of f in D are independent of the integration path.
- If f is analytic in a domain D, containing the loops Γ_1 and Γ_2, and Γ_1 and Γ_2 can be *continuously deformed into each other in D*, then

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$$

- An important integral:

$$\oint_C (z - z_0)^n = 2\pi i \delta_{n,-1}$$

- For any path C that encircles z_0 (once, in the positive direction).

Week 4 – Highlights of last week (2)

Integrals of complex functions

- Cauchy theorem: If f is analytic in a simply connected (no holes) domain D, and C is any closed contour (loop) in D, then

$$\oint_C f(z)dz = 0.$$

- Cauchy integral formula: Let C be a simple, closed, positively oriented contour. If f is analytic in some simply connected domain D containing C, and z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz.$$

- So the behaviour of a function analytic in a domain D is completely determined by the behaviour of f on the boundary of D!

Week 4 – Highlights of last week (3)

Integrals of complex functions

- Generalized Cauchy integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Week 4 – Highlights of last week (4)

Integrals of complex functions

- Cauchy's inequality: Let f be analytic on and inside a circle C_R of radius R centred at z_0. If $|f(z)| \leq M$ for all z on C_R then the derivatives satisfy

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

Theorem

Liouvilles theorem: A function which is analytic and bounded in the entire complex plane is constant.

Week 4 – Highlights of last week (5)

Taylor series

- If f is analytic at z_0 then it can be written in terms of a power series, called Taylor series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

- The disk of convergence is inside the circle around z_0 'touching' the nearest singularity.

Week 4 – Highlights of last week (6)

Laurent series

- If f is analytic in an annulus $r < |z - z_0| < R$ then it can be written as a sum of two series:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} b_k \frac{1}{(z - z_0)^k}$$

- with

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz; \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

- integrated along a closed contour inside the annulus, and encircling z_0.
- The coefficient b_1 is called the *residue* of f at z_0.