Prob. 6:

a) $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$

$\hat{\vec{J}}^2 = \hat{\vec{L}}^2 + 2\hat{\vec{L}}\cdot\hat{\vec{S}} + \hat{\vec{S}}^2$

$\hat{\vec{L}}\cdot\hat{\vec{S}} = \frac{1}{2}\left(\hat{\vec{J}}^2 - \hat{\vec{L}}^2 - \hat{\vec{S}}^2\right)$

b) $\hat{H} = \hat{H}_0 - V_{SO}\,\hat{\vec{L}}\cdot\hat{\vec{S}}$

$\hat{H}_0 |N, l, l\pm 1/2, m_j\rangle = \hbar\omega(N + 3/2)|N, l, l\pm 1/2, m_j\rangle$

$\hat{\vec{L}}\cdot\hat{\vec{S}}|N, l, l\pm 1/2, m_j\rangle$

$= \frac{1}{2}\left(\hat{\vec{J}}^2|N, l, l\pm 1/2, m_j\rangle - \hat{\vec{L}}^2|N, l, l\pm 1/2, m_j\rangle - \hat{\vec{S}}^2|N, l, l\pm 1/2, m_j\rangle\right)$

$= \frac{1}{2}\left((l\pm 1/2)(l\pm 1/2+1)\hbar^2 - l(l+1)\hbar^2 - \frac{3\hbar^2}{4}\right)$

$= \frac{\hbar^2}{2}\begin{cases} l^2 + 2l + \frac{3}{4} - l^2 - l - \frac{3}{4}, & j = l+1/2 \\ l^2 - \frac{1}{4} - l^2 - l - \frac{3}{4}, & j = l-1/2 \end{cases} = \frac{\hbar^2}{2}\begin{cases} l, & j = l+1/2 \\ -l-1, & j = l-1/2 \end{cases}$

$\hat{H}|N, l, l\pm 1/2, m_j\rangle = \left[\hbar\omega\left(N + \frac{3}{2}\right) + \frac{\hbar^2 V_{SO}}{2}\begin{cases} -l, & j = l+1/2 \\ l+1, & j = l-1/2 \end{cases}\right]|N, l, l\pm 1/2, m_j\rangle$

$E_{N,l,j} = \hbar\omega(N + 3/2) + \frac{\hbar^2 V_{SO}}{2}\begin{cases} -l, & j = l+1/2 \\ l+1, & j = l-1/2 \end{cases}$