FYS3500 - spring 2019

Derivation of 2-D Rotation Matrix*

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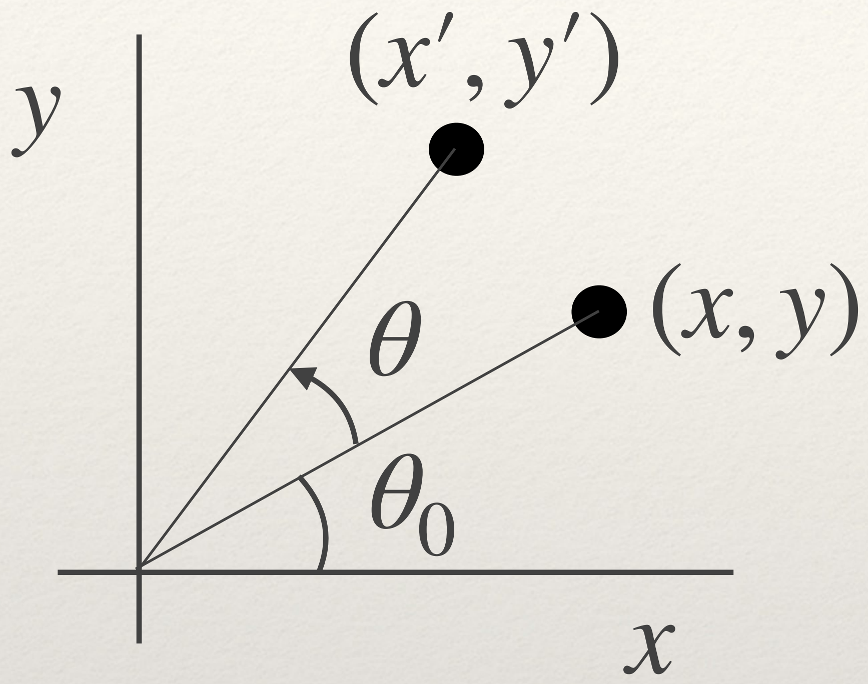
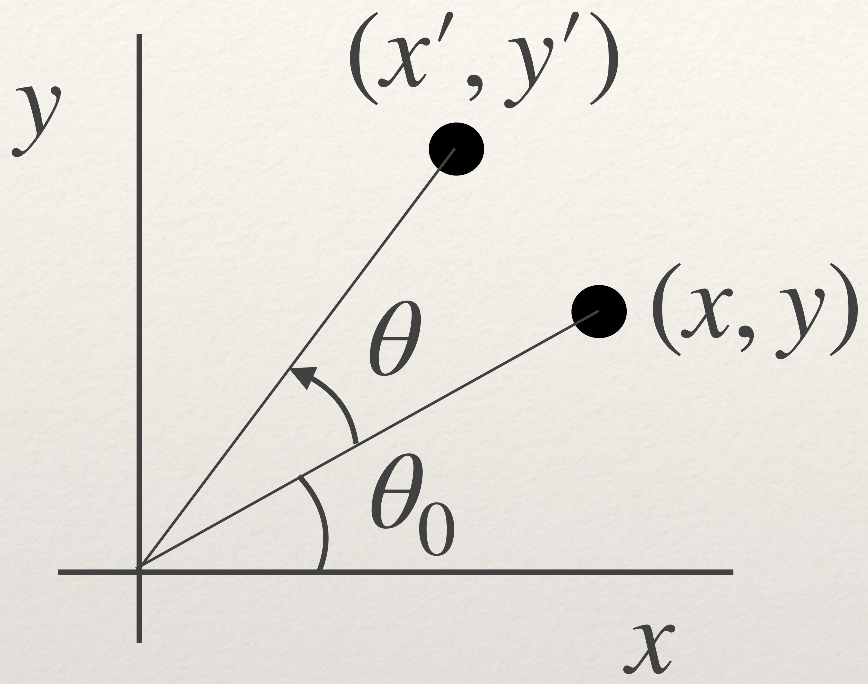
Find rotation operation



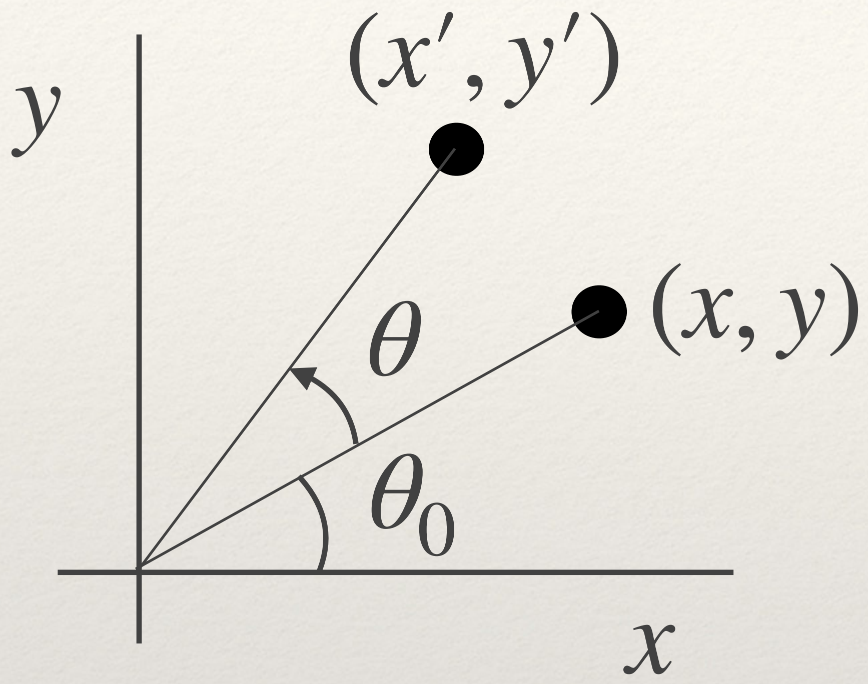
Find rotation operation



Rotation

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Find rotation operation



In r-ϕ coordinates

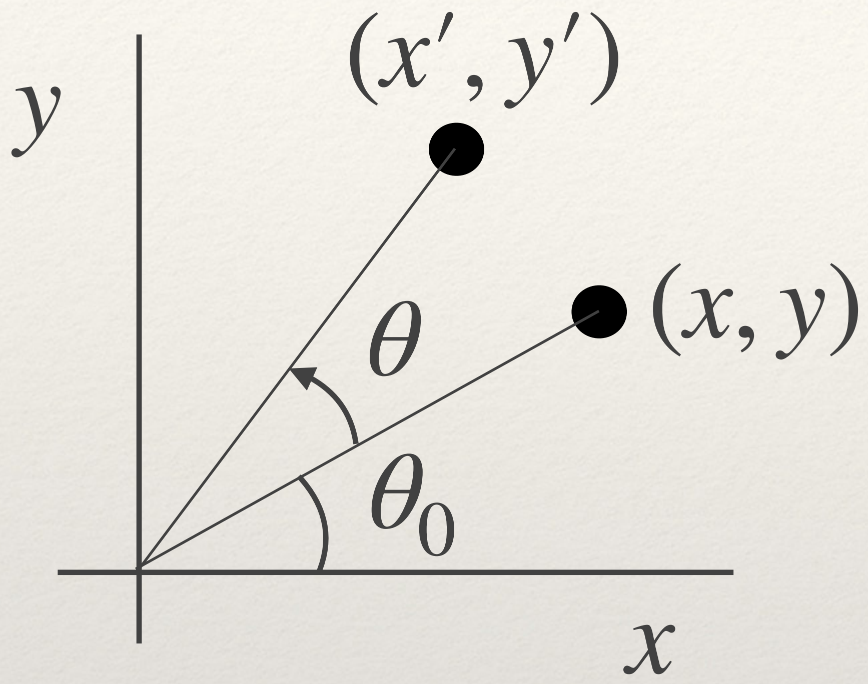
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta_0 \\ r\sin\theta_0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\theta_0 + \theta) \\ r\sin(\theta_0 + \theta) \end{bmatrix}$$

Rotation

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Find rotation operation



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Rotation

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Use identities

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\sin(a \pm b) = \cos a \sin b \pm \sin a \cos b$$

Rotation

Expand with the identities

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\theta_0 + \theta) \\ r\sin(\theta_0 + \theta) \end{bmatrix} = r\begin{bmatrix} \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0 \\ \sin\theta\cos\theta_0 + \cos\theta\sin\theta_0 \end{bmatrix}$$

Rotation

Expand with the identities

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\theta_0 + \theta) \\ r\sin(\theta_0 + \theta) \end{bmatrix} = r\begin{bmatrix} \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0 \\ \sin\theta\cos\theta_0 + \cos\theta\sin\theta_0 \end{bmatrix}$$

Identify x and y

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (r\cos\theta_0)\cos\theta - (r\sin\theta_0)\sin\theta \\ (r\sin\theta_0)\cos\theta + (r\cos\theta_0)\sin\theta \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{bmatrix}$$

Rotation

Expand with the identities

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\theta_0 + \theta) \\ r\sin(\theta_0 + \theta) \end{bmatrix} = r\begin{bmatrix} \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0 \\ \sin\theta\cos\theta_0 + \cos\theta\sin\theta_0 \end{bmatrix}$$

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Matrix equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation

Expand with the identities

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\theta_0 + \theta) \\ r\sin(\theta_0 + \theta) \end{bmatrix} = r\begin{bmatrix} \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0 \\ \sin\theta\cos\theta_0 + \cos\theta\sin\theta_0 \end{bmatrix}$$

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Matrix equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Inverse rotation

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

Inverse rotation

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotate back

$$R^{-1}(\theta) \begin{bmatrix} x' \\ y' \end{bmatrix} = R^{-1}(\theta) R(\theta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Inverse rotation

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotate back

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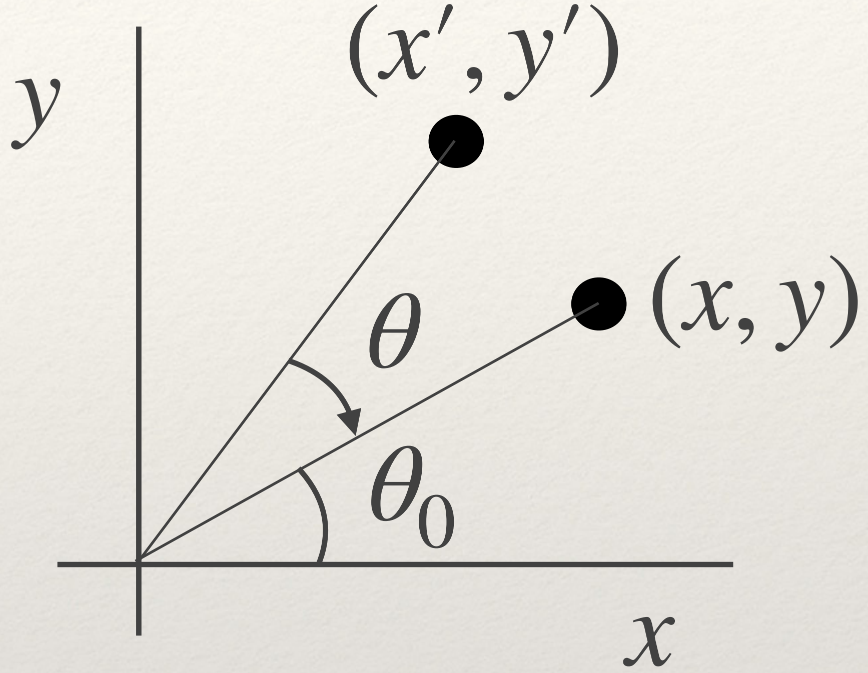
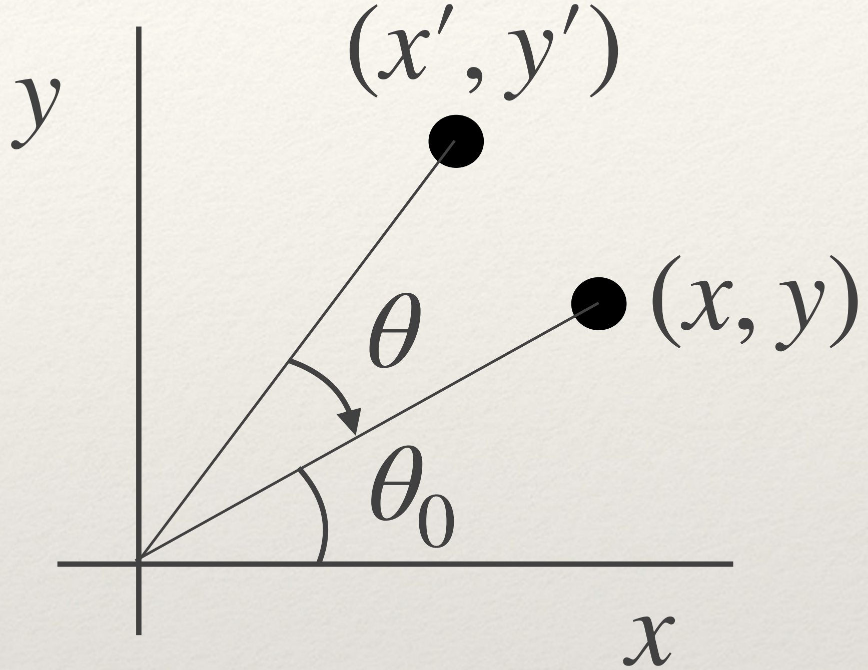


Inverse rotation

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotate back

$$R^{-1}(\theta) \begin{bmatrix} x' \\ y' \end{bmatrix} = R^{-1}(\theta) R(\theta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Recall

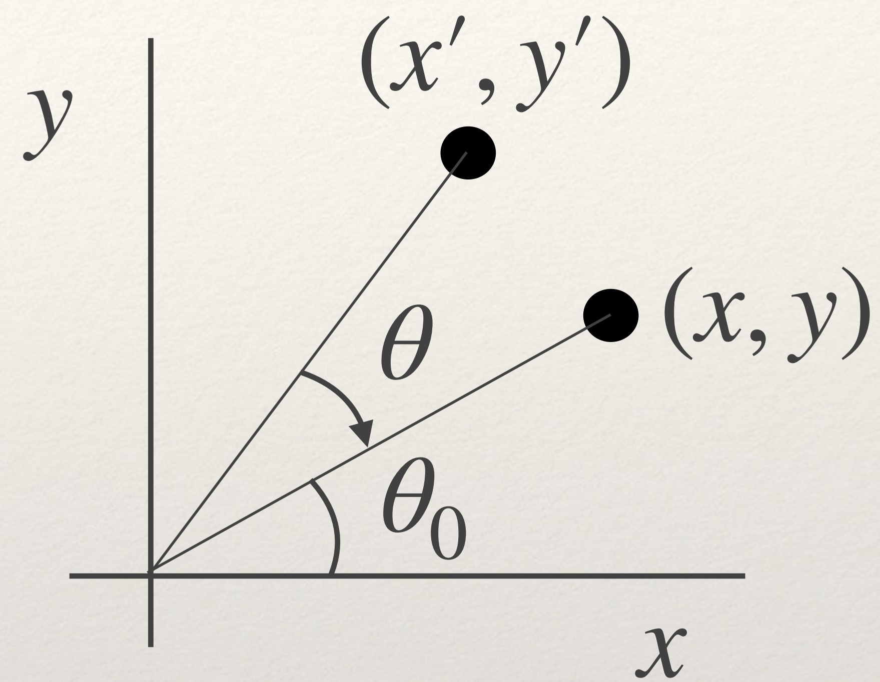
$$\cos(-\theta) = \cos(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

Inverse rotation

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotate back

$$R^{-1}(\theta) \begin{bmatrix} x' \\ y' \end{bmatrix} = R^{-1}(\theta) R(\theta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Recall

$$\cos(-\theta) = \cos(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

Inverse rotation

$$R^{-1}(\theta) = R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$