FYS3500 - Problem set 2

Spring term 2019

Problem 1 – In class

a) What is the nuclear radius? Give several different ways to measure it and compare important differences.

b) All known heavy nuclei exhibit an excess neutrons. Looking eg. at ^{240}Pu, it has more then 40 % more neutrons then protons. Naively one should expect a larger radius for the neutrons then for the protons ($\rightarrow$ How do we usually call these radii), however, they are observed to agree within about 0.1fm. How can this be explained?

c) When analyze the scattering of α-particles on medium and heavy nuclei we can learn about the nuclear radius. How?

d) What is the angular dependence of scattered α-particles? Sketch the cross-section of backscattered nuclei (scattering angle $\theta \approx \pi$) as a function of the incident energy. Take this to estimate the nuclear radius of ^{64}Cu, where the critical energy E_c is about 13.7 MeV.
Hint: What is the relation of kinetic and potential energy in backscattering?

Problem 2 – in class

When we describe nuclear γ-ray resonances, we usually give the energy E_γ of an emitted photon is the difference E_0 between the excited state with energy E_x and the ground state (GS), $E_\gamma = E_0 = E_\text{x} - E_\text{GS}$. This is not exact for free atoms or molecules, as it neglects the recoil energy of the nucleus.

a) Suppose that the nucleus was at rest before γ-ray emission and calculate the exact gamma-ray energy. Can we neglect the recoil effect?

b) ^{60}Co is one of the most important γ-ray calibration sources. By β-decay it feeds excited levels in ^{60}Ni. The 1332 keV level (with direct decay into ground state) has a half-life $t_{1/2}$ below 1ps ($< 10^{-12}$ s). What does this imply for the energy of the γ-ray emitted in the emission? (Remember to convert half-life to lifetime)

c) What are the implication for the nuclear resonance absorption of γ-ray photons? Argue qualitatively how these results would be effected if the nucleus was not at rest during decay.

d) (At home?) What is the Mössbauer effect and how does this combine with these results?

Problem 3: Nuclear binding energy

The emiprical mass formula is (neglecting the odd-even effect)

$$M(A,Z) = Z(m_p + m_e) + (A-Z)m_n - a_v A + a_s A^{2/3} + a_c \frac{Z(Z-1)}{A^{1/3}} + a_\text{sym}\frac{(A-2Z)^2}{A} \quad (1)$$

Where a_v, a_s, a_c and a_sym are constants. Explain the dependence on Z and A for the different terms.

a) Two isobar nuclei 1 and 2 are called mirror nuclei if they result in each other by the exchange of the neutron an proton number: $A_1 = A_2, Z_1 = N_2, N_1 = Z_2$. They have a analogous structure and therefore also the same quantum numbers for the total angular momentrum J and parity π. As an example we look at ^{27}Al and ^{27}Si, both having (J=5/2,π=+1) in the ground state. ^{27}Si decays by β^+-decay into ^{27}Al ($^{27}\text{Si} \rightarrow {}^{27}\text{Al} + e^+ + \nu$), where the sum of the kinetic energies of the positron and neutrino has been measured to be at max $E_0 = 3.8$ MeV. Deduce the radius parameter r_0 from E_0.

Hint: If we assume that the nuclear forces are charge independent, the energy released in the decay of mirror nuclei is given by the mass difference $(m_n - m_p)^2 = 1.29$ MeV and the difference in the coulomb energy. The charge distribution can be assumes as a homogeneously charged sphere (total charge Ze and radius R) with the coulomb energy W_c

$$W_c = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0}\frac{1}{R} \tag{2}$$

b) Use (1) to show that for a constant A the Z value that corresponds to the most stable nucleus is

$$Z_{\text{min}} = \frac{m_n c^2 - (m_p + m_e)c^2 + a_c A^{-1/3} + 4a_{\text{sym}}}{2a_c A^{-1/3} + 8a_{\text{sym}} A^{-1}}$$

c) Determine the most stable isobar with mass number $A = 87$. (Again, use $a_v = 15.5$ MeV, $a_s = 16.8$ MeV, $a_c = 0.72$ MeV and $a_{\text{sym}} = 23$ MeV).

Problem 4: Reaction Q-values

Find the Q-value of the reaction:

$$^{86}\text{Kr} + \text{d} \rightarrow ^{87}\text{Kr} + \text{p}$$

Rember: d = ^{2}H.

Problem 5: "The binding energy plot"

Explain how the mass of a nucleus can be calculated from the plot in figure below. Explain briefly some of the main features of the plot and estimate the mass of ^{130}Xe. What is the relation to fission and fusion? How did we get the heavy elements?

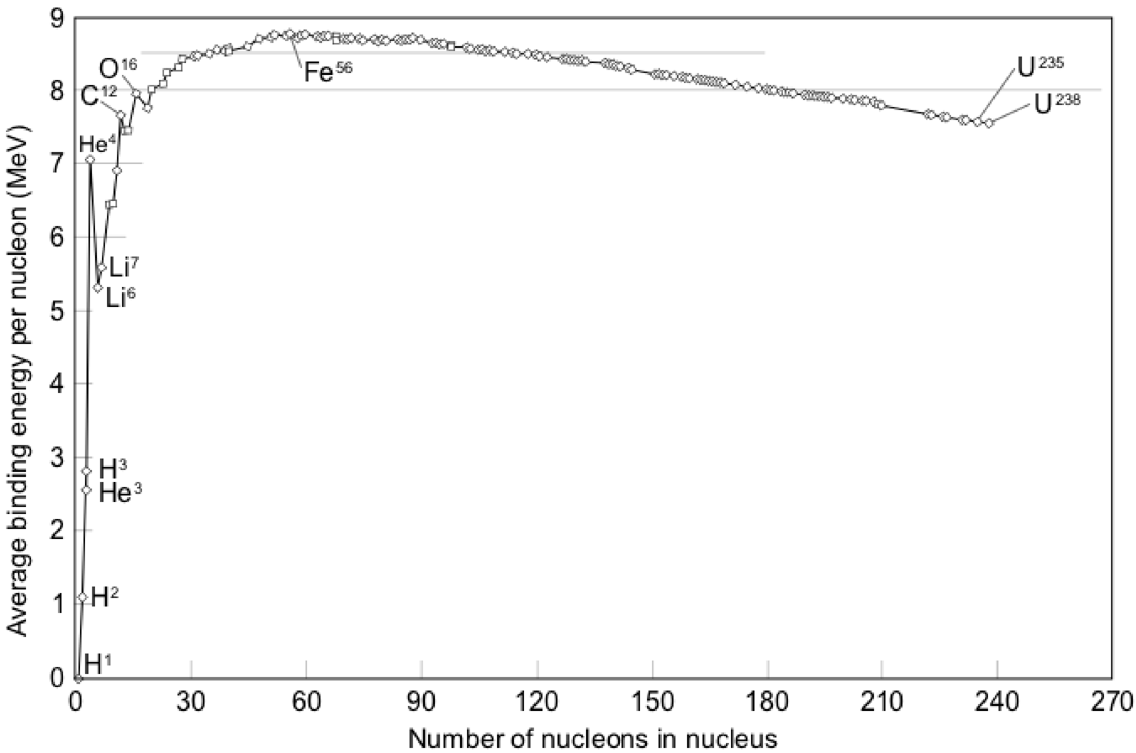

Figure 1: Average binding energy per nucleon for stable nuclei as a function of the mass number.

Problem 6: Nuclear Scattering

In Figure 2 the differential cross-sections $d\sigma/d\Omega$ for scattering of high energy electrons on ^{40}Ca and ^{48}Ca are displayed.

a) Explain the general behavior of the cross-section as a function of the scattering angle. What is the source for the (local)minima?

b) The form factor $F(\boldsymbol{q}^2)$ is defined as the Fourier transformation of the charge density $\rho(\boldsymbol{r})$, with the momentum transfer $\boldsymbol{q} = \boldsymbol{k}_\mathrm{f} - \boldsymbol{k}_\mathrm{i}$ and the initial and final momentum $\boldsymbol{k}_{\mathrm{i,f}}$. Under the usual assumption of the Born approximation and negligible recoil, $F(\boldsymbol{q}^2)$ can be calculated by

$$F(\boldsymbol{q}^2) = \int e^{i\boldsymbol{q}\boldsymbol{r}/\hbar}\rho(\boldsymbol{r})\mathrm{d}^3\boldsymbol{r}. \tag{3}$$

Calculate $F(\boldsymbol{q}^2)$ for a *homogeneous* spherical charge distribution, $\rho(r) = 3Ze/(4\pi R^3)$ for $r < R$, and 0 elsewhere. Show that the result is given by

$$F(\boldsymbol{q}^2) = 3Ze(\frac{\hbar}{qR})^3[(\frac{qR}{\hbar})\cos(\frac{qR}{\hbar}) - \sin(\frac{qR}{\hbar})] \tag{4}$$

c) Plot the result. (It is important to choose a reasonable scaling for q. Recall that $|\boldsymbol{q}| = 2|\boldsymbol{p}|\sin(\frac{\theta}{2})$)

d) Use this and Figure 2 to compare the radius of ^{40}Ca and ^{48}Ca. Hint: How does the location of the minima depend on R for given angle.

e) Some extra calculations ("bonus"): Develop eq.(3) for a general spherical potential in powers of $|\boldsymbol{q}|$ (first 2 non-vanishing terms) using the mean square radius $\langle r^2 \rangle$. This will lead you to

$$F(\boldsymbol{q}) = 1 - \frac{1}{6} \frac{\boldsymbol{q} \langle r^2 \rangle}{\hbar^2} + \ldots . \tag{5}$$

How can you solve the expression for the mean square charge radius $\langle r^2 \rangle$. What momentum transfers are most important in order to measure $\langle r^2 \rangle$? What does this mean for the experiment (for example, would you measure at certain angles, rather high or low energies,...?) Hint: a) The mean square charge radius is defined as $\langle r^2 \rangle = \int r^2 \rho(r)\, d^3r$. b) Remember that the form factor is normed such that $F(\boldsymbol{q}^2 = 0) = 0$.

f) *For further thought ;P*: Experimentally only a restricted range of momentum transfers is accessible, as it is limited by the beam intensity. In addition, the cross-section drops quickly with increasing momentum transfers. Think about a method to determine the charge distribution despite these problems.

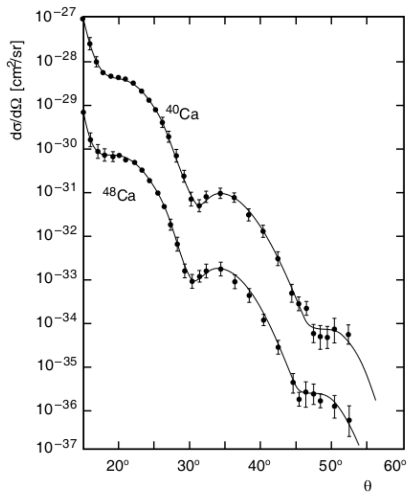

Figure 2: Differential cross-section for scattering of 750 MeV electrons on ^{40}Ca and ^{48}Ca. The cross-sections have been multiplied by 10 for ^{40}Ca and by 10^{-1} for ^{48}Ca for displaying purposes. Source: J. B. Bellicard, et al. Phys. Rev. Lett. **19** (1967) 527

Problem 7: LS-coupling in the shell model – Special hand-in for this part: In 2 weeks

In order to get the correct magic numbers in the nuclear shell model we need to include a spin-orbit coupling. This leads to a Hamiltonian similar to:

$$\hat{H} = \hat{H}_0 - V_{\mathrm{SO}} \hat{\vec{L}} \cdot \hat{\vec{S}} \tag{6}$$

Where $\hat{H}_0$ is a Hamiltonian with eigenstates $\hat{H}_0 |N, l\rangle = \hbar\omega(N + 3/2) |N, l\rangle$ with $l = N, N - 2, N - 4, \ldots 1$ or $0, l \geq 0$ and $V_{\mathrm{SO}} \hat{\vec{L}} \cdot \hat{\vec{S}}$ is the spin-orbit coupling. V_{SO} is the strength of the coupling and can be regarded as a constant in this problem. In this problem we only look at spin-1/2 fermions.

a) The operator for the total spin is $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$. Where $\hat{\vec{L}}$ is the angular momentum operator and $\hat{\vec{S}}$ is the spin operator. Show that:

$$\hat{\vec{L}} \cdot \hat{\vec{S}} = \frac{1}{2}(\hat{\vec{J}}^2 - \hat{\vec{L}}^2 - \hat{\vec{S}}^2)$$

b) For a given angular momentum l the total spin state $|j, m_j\rangle$ can be $j = |l - 1/2|, l + 1/2$. The two possible states can be written as a superposition of $|l, m_l = m_j \pm 1/2\rangle \otimes |s = 1/2, m_s \mp 1/2\rangle$ states:

$$\begin{aligned} |j = l \pm 1/2, m_j\rangle = C^l_{1/2} \, |l, m_j - 1/2\rangle \otimes |\uparrow\rangle \\ + C^l_{-1/2} \, |l, m_j + 1/2\rangle \otimes |\downarrow\rangle \,, \end{aligned}$$

where the Clebsch-Gordan coefficients $C^l_{m_s} = 0$ if $|m_j - m_s| > l$ or $m_j + m_s > l$; and $|\uparrow\rangle = |1/2, 1/2\rangle$, $|\downarrow\rangle = |1/2, -1/2\rangle$. Find the energy of the state $|N, l, j = l \pm 1/2, m_j\rangle$ using (6).