Problem Set 1

1|

a) We use $\vec{L} = \vec{r} \times \vec{p} \rightarrow \hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$

so $L_x = y p_z - z p_y$ [dropping $\hat{}$ hat for convenience]

$L_y = z p_x - x p_z$

as $[a, b] = ab - ba$

$$\rightarrow [L_x, L_y] = y p_z z p_x - \cancel{y p_z x p_z} - \cancel{z p_y z p_x} + z p_y x p_z$$
$$- (z p_x y p_z - \cancel{z p_x z p_y} - \cancel{x p_z y p_z} + x p_z z p_y)$$

since $[r_i, p_j] = i\hbar \delta_{ij}$

$z p_z - p_z z = i\hbar$

$$= \cancel{y p_z z p_x} - (i\hbar + \cancel{p_z z}) y p_x + \cancel{z p_y x p_z} - (z p_z - i\hbar) x p_y$$

$$= -i\hbar y p_x + i\hbar x p_y = i\hbar L_z$$

Significance of $[L_x, L_y] \neq 0$: Can't simultaneously measure precisely two angular momentum components. Analog for $[L_x, y]$ and $[L_x, p_y] \neq 0$

$[L^2, L_x] = 0 \Rightarrow$ System of good quantum numbers; describing a state by L^2 and one component (eg. L_x)

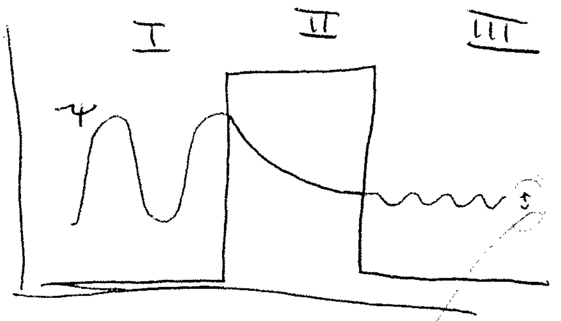
b) i) $L = 0$, $S = \frac{1}{2}$ $\rightarrow$ $\vec{J} = |\vec{L} + \vec{S}| = \frac{1}{2}$; $m_j = -\frac{1}{2}, \frac{1}{2}$

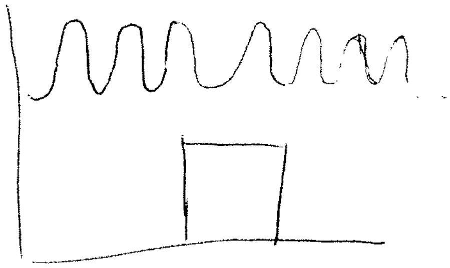
${}^2S_{\frac{1}{2}}$ with #deg 2

ii) $L = 2$, $S = \frac{1}{2}$ $\rightarrow$ $\vec{J} = \frac{3}{2}$, $m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ $\xrightarrow{\#deg=4}$ ${}^2d_{3/2}$

$\vec{J} = \frac{5}{2}$, $m_j = -\frac{5}{2}, \ldots, \frac{5}{2}$ $\xrightarrow{\#deg=6}$ ${}^2d_{5/2}$

Problem 2 |





I ⇒ also reflection

II ⇒ exp. decay

$\psi_{III} \neq 0$
⇒ tunnelling
eg. in α-decay
(& fission)

larger/wider barrier
⇒ smaller amplitude

Slightly incorrect drawing here: Th

$A\, e^{ik_I x}$ →

← $B e^{-ik_I x}$

with $B \neq 0$
even for
$E > V_{II}$!

⇒ ≠ classical case

⇒ partly (destructive!) interference

P3 |

a) $R(kr) \sim j_\ell(kr)$ — Bessel-fkt. dependent on ℓ

k: wavenumber $\sim n$

arrises due to boundary conditions!

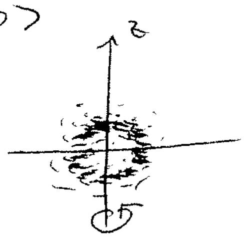
$|\psi(r=R)| = 0$!

Energy depends on n and l, but not on m_l ($=m$).
Thus degeneracy ⇒ $2\ell + 1$

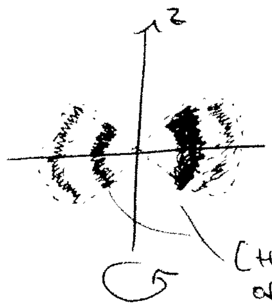
b) $\ell \leq n$, $|m| \leq l$

c) given in the picture: $|r R(r)|^2$ and $|Y_\ell^m|^2$

$|0,0,0\rangle$



$|2,1,1\rangle$



[these two should of course be the same!]

Problem 4

a) $^{232}_{90}Th_{142}$, $^{235}_{92}U_{143}$

b) ^{20}Na

c) ^{206}Pb, ^{207}Pb, ^{208}Pb

d) —

e) $^{90-92,94}Zr$, (^{96}Zr)
$T_{1/2} \approx 10^{19}$ years

P5

a)

$^{16}_{8}O_{8}$	$^{19}_{10}Ne_{9}$	$^{18}_{9}F_{9}$	$^{15}_{7}N_{8}$	$^{16}_{6}C_{10}$
$^{20}_{8}O_{12}$	$^{16}_{10}Ne_{6}$		$^{20}_{7}N_{13}$	$^{14}_{6}C_{8}$
	$^{20}_{10}Ne_{10}$			

a) ordered by isotopes

b) isotones $N=8$: ^{16}O, ^{15}N, ^{14}C

$N=9$: F_9, Ne_9

$N=10$: C_{10}, Ne_{10}

c) isobars $A=16$: ^{16}O, ^{16}Ne, ^{16}C

$A=20$: ^{20}N, ^{20}O, ^{20}Ne

note that $^{16}_{6}C_{10}$ and $^{16}_{10}Ne_{6}$ are called mirror nuclei

e) isomer = excited state of nuclei with long lifetime

P6|

	^{12}C	^{239}Pu
a) [u]	12	~ 239.05
b) $[MeV/c^2]$	~ 11178	~ 22267
c) [kg]	$\sim 1.99 \times 10^{-26}$	$\sim 3.97 \times 10^{-25}$

$1u = 931{,}502 \frac{MeV}{c^2}$

$1 \left[\frac{MeV}{c^2}\right] = 1.783 \cdot 10^{-30} kg$

P7|

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\,\psi(x)$$

a)

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x,t)\right)\psi(x,t) = i\hbar \frac{\partial}{\partial t}\psi(x,t)$$

with separation $\psi(x,t) = \psi(x)\cdot\psi(t)$ and ansatz $\psi(t) = e^{-i\frac{E}{\hbar}t}$

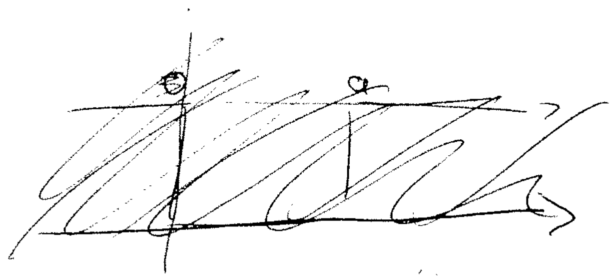
we find

$$\cancel{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x,t) = i\hbar\,\psi(x)\left(-\tfrac{i}{\hbar}E\right)\psi(t)$$

$$= \cancel{i\hbar}\, E\,\psi(x,t)$$

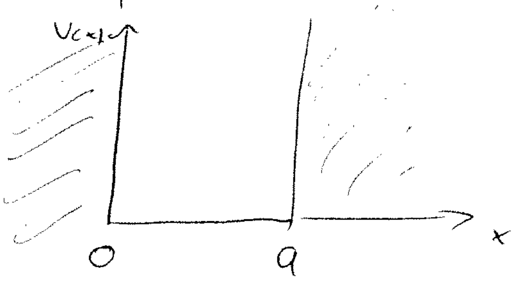
$$\rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\,\psi(x)$$

b) infinite well:



boundary conditions:

i) $\psi(0) = \psi(x=a) = 0$



Ansatz: $\psi = A\sin(kx) + B\cos(kx)$ as solutions to SGL

$\Rightarrow$ continuity $\rightarrow$ can't be cosine ($\cos(k\cdot 0) \neq 0$)

$\psi(x=a) = A\sin(ka) = 0$

$\Rightarrow ka = n\pi$ with $n = 1, 2, 3, \ldots$

$\rightarrow$ plugg in SGL: $E_n = \frac{\hbar k^2}{2m} \left(= \left(\frac{\hbar\pi}{a}\right)^2 \frac{1}{2m} n^2\right.$

Thus, the normalized wave-function is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

c) We now assume that $\varphi(x) = A\left(\psi_1(x) + \psi_3(x)\right)$

We need to find A:

$$\int_0^a |A|^2 \left(|\psi_1(x)|^2 + \psi_1^*(x)\psi_3(x) + \psi_3^*(x)\psi_1(x) + |\psi_3(x)|^2\right) dx$$

$$= |A|^2 \int_0^a \left(|\psi_1(x)|^2 + |\psi_3(x)|^2 + \psi_1^*\psi_3 + \psi_3^*\psi_1\right) dx$$

* Due to orthogonality, $\int_0^a \psi_n^* \psi_m \, dx = \delta_{nm}$

$$\int_0^a |\varphi|^2 dx = |A|^2 \cdot 2 = 1 \Rightarrow |A| = \frac{1}{\sqrt{2}}$$

$$\varphi(x) = \frac{\psi_1(x) + \psi_2(x)}{\sqrt{2}}$$

d)

We want to find the average energy of ψ:

$$\hat{H}\psi = \frac{1}{\sqrt{2}}\left(\hat{H}\psi_1 + \hat{H}\psi_3\right) = \frac{1}{\sqrt{2}}\left(\frac{\pi^2\hbar^2}{2ma^2}\psi_1 + \frac{9\pi^2\hbar^2}{2ma^2}\psi_3\right)$$

$$= \frac{\pi^2\hbar^2}{2ma^2}\frac{1}{\sqrt{2}}\left(\psi_1 + 3\psi_3\right)$$

$$\langle E \rangle = \int_0^a \psi^* \hat{H} \psi \, dx = \frac{\pi^2\hbar^2}{2ma^2}\frac{1}{2}\left(1 + 9\right) = \frac{5\pi^2\hbar^2}{2ma^2}$$

$$\langle E^2 \rangle = \int_0^a \psi^* \hat{H}^2 \psi \, dx =$$

$$\hat{H}\hat{H}\psi = \frac{\pi^2\hbar^2}{2ma^2}\frac{1}{\sqrt{2}}\left(\hat{H}\psi_1 + 9\hat{H}\psi_3\right)$$

$$= \frac{\pi^2\hbar^2}{2ma^2}\frac{1}{\sqrt{2}}\left(\frac{\pi^2\hbar^2}{2ma^2}\psi_1 + 81\frac{\pi^2\hbar^2}{2ma^2}\psi_3\right)$$

$$= \left(\frac{\pi^2\hbar^2}{2ma^2}\right)^2 \frac{1}{\sqrt{2}}\left(\psi_1 + 81\psi_3\right)$$

$$\langle E^2 \rangle = \left(\frac{\pi^2\hbar^2}{2ma^2}\right)^2 \frac{82}{2} = 41\left(\frac{\pi^2\hbar^2}{2ma^2}\right)^2$$

$$(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2 = 16\left(\frac{\pi^2\hbar^2}{2ma^2}\right)^2$$

$$\Delta E = 4\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$$

09.35
10.27

1) Propabillity of finding $\varphi(x)$ between $\frac{a}{4}$ and $\frac{3a}{4}$

$$\varphi(x) = \sqrt{\frac{2}{a}}\,\frac{\sin\left(\frac{\pi}{a}x\right)+\sin\left(\frac{3\pi}{a}x\right)}{\sqrt{2}} = \frac{\sin\left(\frac{\pi}{a}x\right)+\sin\frac{3\pi}{a}x}{\sqrt{a}}$$

$$P\left(\frac{a}{4}\le x\le\frac{3a}{4}\right) = \int_{a/4}^{3a/4}|\Psi(x)|^2\,dx$$

$$= \int_{a/4}^{3a/4}\frac{1}{a}\left(\sin\left(\frac{\pi}{a}x\right)+\sin\left(\frac{3\pi}{a}x\right)\right)^2 dx$$

$$= \frac{1}{a}\int_{a/4}^{3a/4}\sin^2\left(\frac{\pi}{a}x\right)dx + \frac{2}{a}\int_{a/4}^{3a/4}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{3\pi}{a}x\right)dx + \frac{1}{a}\int_{a/4}^{3a/4}\sin^2\left(\frac{3\pi}{a}x\right)dx$$

First we solve $\int_{a/4}^{3a/4}\sin^2\left(\frac{\pi}{a}x\right)dx$.

We substitute $u=\frac{\pi}{a}x$, $du=\frac{\pi}{a}dx$, $u(a/4)=\frac{\pi}{4}$, $u\left(\frac{3a}{4}\right)=\frac{3\pi}{4}$

$$\int_{a/4}^{3a/4}\sin^2\left(\frac{\pi}{a}x\right)dx = \frac{a}{\pi}\int_{\pi/4}^{3\pi/4}\sin^2 u\,du = \frac{a}{2\pi}\Big[u-\sin u\cos u\Big]_{\pi/4}^{3\pi/4}$$

$$= \frac{a}{2\pi}\left[\frac{3\pi}{4}-\frac{\pi}{4}+\frac{1}{2}+\frac{1}{2}\right] = \frac{a}{2\pi}\left[\frac{\pi+2}{2}\right] = \frac{(2+\pi)a}{4\pi}$$

For $\int_{a/4}^{3a/4} \sin^2\left(\frac{3\pi}{a}x\right) dx$ we let

$u = \frac{3\pi}{a}x$, $du = \frac{3\pi}{a}dx$, $u(a/4) = \frac{3\pi}{4}$, $u\left(\frac{3a}{4}\right) = \frac{9\pi}{4}$

$$\int_{a/4}^{3a/4} \sin^2\left(\frac{3\pi}{a}x\right) dx = \frac{a}{3\pi}\int_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} \sin^2 u \, du$$

$$= \frac{a}{6\pi}\Big[u - \sin u \cos u\Big]_{3\pi/4}^{9\pi/4}$$

$$= \frac{a}{6\pi}\left[\frac{9\pi - 3\pi}{4} - \frac{1}{2} - \frac{1}{2}\right] = \frac{a}{6\pi}\left[\frac{3\pi - 2}{2}\right]$$

$$= \frac{(3\pi - 2)a}{12\pi}$$

For $\int_{a/4}^{3a/4} \sin\frac{\pi}{a}x \sin\frac{3\pi}{a}x \, dx$ we use that

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

So

$$\int_{a/4}^{3a/4} \sin\left(\frac{\pi}{a}x\right)\sin\frac{3\pi}{a}x \, dx = \frac{1}{2}\int_{a/4}^{3a/4} \cos\left(-\frac{2\pi}{a}x\right) - \cos\left(\frac{4\pi}{a}x\right) dx$$

$\cos(-x) = \cos(x)$

$$= \frac{1}{2}\int_{a/4}^{3a/4} \cos\left(\frac{2\pi}{a}x\right) - \cos\left(\frac{4\pi}{a}x\right) dx$$

$$= \frac{1}{2}\left[\frac{a}{2\pi}\sin\left(\frac{2\pi}{a}x\right) - \frac{a}{4\pi}\sin\left(\frac{4\pi}{a}x\right)\right]_{a/4}^{3a/4}$$

$$= \frac{1}{2}\left[\frac{a}{2\pi}\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1} - \frac{a}{2\pi}\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} - \frac{a}{4\pi}\underbrace{\sin 3\pi}_{=0} + \frac{a}{4\pi}\underbrace{\sin \pi}_{=0}\right]$$

$$= -\frac{2}{2}\frac{a}{2\pi} = -\frac{a}{2\pi}$$

So:

$$P\left(\frac{a}{4} \le x \le \frac{3a}{4}\right) = \int_{a/4}^{3a/4} |\psi|^2 dx$$

$$= \frac{1}{a}\int_{a/4}^{3a/4} \sin^2 \frac{\pi}{a}x\, dx + \frac{2}{a}\int_{a/4}^{3a/4} \sin\frac{\pi}{a}x \sin\frac{3\pi}{a}x\, dx + \frac{1}{a}\int_{a/4}^{3a/4} \sin^2 \frac{3\pi}{a}x\, dx$$

$$= \frac{1}{\cancel{a}}\frac{(2+\pi)}{4\pi}\cancel{a} - \frac{2}{\cancel{a}}\frac{a}{2\pi} + \frac{1}{\cancel{a}}\frac{(3\pi-2)\cancel{a}}{12\pi}$$

$$= \frac{2}{4\pi} + \frac{\pi}{4\pi} - \frac{1}{\pi} + \frac{3\pi}{12\pi} - \frac{2}{12\pi}$$

$$= \frac{1}{2\pi} + \frac{1}{4} - \frac{1}{\pi} + \frac{1}{4} - \frac{1}{6\pi}$$

$$= \frac{1}{2} + \frac{3}{6\pi} - \frac{6}{6\pi} - \frac{1}{6\pi} = \frac{1}{2} - \frac{4}{6\pi} = \frac{1}{2} - \frac{2}{3\pi} \approx \underline{\underline{0,29}}$$

9)

a) $\hat{a} = \sqrt{\frac{1}{2\hbar m\omega}} \, (m\omega\hat{x} + i\hat{p})$

$a^+ = \sqrt{\frac{1}{2\hbar m\omega}} \, (m\omega\hat{x} - i\hat{p})$

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

~~$\hat{a}^+|n\rangle$~~ $\hat{a}^+|n-1\rangle = \sqrt{\underbrace{(n-1)+1}_{=n}}\,|n\rangle$

$\to \hat{a}^+\hat{a}\,|n\rangle = n|n\rangle$

$\Rightarrow \underline{\underline{\hat{a}^+\hat{a} = \hat{N}}}$

$\Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2}\frac{1}{m\omega}} \, (a^+ + a)$

$\hat{p} = i\sqrt{\frac{\hbar}{2} m\omega} \, (a^+ - a)$,

and $H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$

$\hat{p}^2 = -\frac{\hbar}{2} m\omega \left(\hat{a}^{+2} - \hat{a}^+\hat{a} - \hat{a}\hat{a}^+ + \hat{a}^2 \right)$

with $[\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$

$= -\frac{\hbar}{2} m\omega \left(\hat{a}^{+2} + \hat{a}^2 - \hat{a}^+\hat{a} - (1 + \hat{a}^+\hat{a}) \right)$

$= -\frac{\hbar}{2} m\omega \left(\hat{a}^{+2} + \hat{a}^2 - 2\hat{N} - 1 \right)$

$\hat{x}^2 = \ldots = \frac{\hbar}{2}\frac{1}{m\omega} \left(\hat{a}^{+2} + \hat{a}^2 + 2\hat{N} + 1 \right)$

$\hat{H} = \ldots = \hbar\omega \left(\hat{N} + \frac{1}{2} \right)$

$\Rightarrow \underline{\hat{H}|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right)}$

b) i) $\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[\psi_0(x_1)\,\psi_1(x_2) - \psi_0(x_2)\,\psi_1(x_1) \right]$

ii) $= \frac{1}{\sqrt{2}} \left[\psi_0(x_1)\,\psi_2(x_2) - \psi_0(x_2)\,\psi_2(x_1) \right]$

iii) $= \frac{1}{\sqrt{2}} \left[\psi_1(x_1)\,\psi_2(x_2) - \psi_1(x_2)\,\psi_2(x_1) \right]$

$\to$ can't have two ~~particle~~ fermions in same state (i... spin) !

c) degeneracy

$$E = E_x + E_y + E_z = \hbar\omega\left(n_x + n_y + n_z + \frac{1}{2}\times 3\right)$$

$\Rightarrow$ with $E = \frac{7}{2}\hbar\omega$ we find

$|n_x, n_y, n_z\rangle$

$$\left.\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array}\right\} \text{6 degenerate states}$$