FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2004

The problem set is available from Friday October 15. The set consists of 2 problems written on 5 pages.

Deadline for returning solutions
is Friday October 22.

Return of solutions
The solutions can be returned either in written/printed form or as an e-mail attachment.
Written/printed solutions can be returned at Ekspedisjonskontoret in the Physics Building. Please add a copy that the lecturer can keep for evaluation at the final exam.
E-mailed solutions: Please send the solutions as one file, preferably in pdf format. E-mail address:j.m.leinaas@fys.uio.no.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Auditorium or Office 471); Monday morning available between 9 and 10 a.m.

Language
Solutions may be written in Norwegian or English, depending on your preference.

PROBLEMS

1 Particle encircling a magnetic flux
A particle with mass m and charge e moves freely on a circle of radius R. Through the circle passes a solenoid that carries a magnet flux Φ. We may consider the total flux to be confined to the solenoid so that the magnetic field vanishes on the circle where the particle moves.

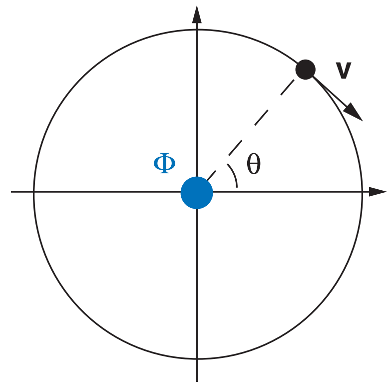
In the following make use of the general expressions for the Hamiltonian of a particle in a magnetic field

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 \tag{1}$$

and for the probability current

$$\mathbf{j} = -\frac{i\hbar}{2m}(\psi^* \boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*) - \frac{e}{mc}\mathbf{A}\psi^*\psi \tag{2}$$

Use the angle variable θ as coordinate for the particle on the circle.



a) Assume rotational invariance about the center of the circle and show that the vector potential on the circle takes the constant value $A = \Phi/2\pi R$ with direction along the circle. Explain why this vector potential has no influence on the motion of the particle when this is described by the classical equations of motion.

b) Express the Hamiltonian as an operator acting on the wave functions $\psi(\theta)$ for the particle on the circle. Find the energy eigenvalues and show that the energy spectrum varies periodically with the flux Φ. What is the flux period Φ_0? Plot the four lowest energies as functions of Φ in the interval from 0 to Φ_0. Characterize the ground state by its angular momentum in the same interval. What is special for the spectrum at $\Phi = \Phi_0/2$?

c) Find the probability current for a general wave function $\psi(\theta)$, and determine the value of the ground state current as a function of Φ. What is the maximum value of the ground state current and what value for the particle velocity does that correspond to.

d) Find the propagator $\mathcal{G}(\theta, t; 0, 0) = \langle \theta, t | 0, 0 \rangle$ expressed in terms of the Jacobi theta function for general Φ. Use the definition of the Jacobi theta function as given in problem 2.4 (Problem Set 2).

e) For the Lagrangian of a particle in a magnetic field the effect of the vector potential is to add a term proportional to the velocity

$$L = \frac{1}{2} m v^2 + \frac{e}{c} \mathbf{A} \cdot \mathbf{v} \tag{3}$$

Follow the path integral approach of problem 2.4 (Problem Set 2) to find the propagator by summing over all classical paths with the given initial and final conditions. Show in the same way as discussed there that the propagator derived in this way is equivalent to the one derived in d). (Use the properties listed in Problem 2.4 for the Jacobi theta function.)

2 Entangled photons

In this problem correlations between pairs of entangled photons are studied. The interesting degree of freedom is the polarization of each photon. For a single photon this means that the quantum state is a vector in a two-dimensional vector space spanned by the vectors $|H\rangle$ and $|V\rangle$, which correspond to linear polarization in the horizontal and vertical direction, respectively. A general polarization state is a linear combination of these two. As special cases we consider linearly polarized photons in a rotated direction,

$$|\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle \tag{4}$$

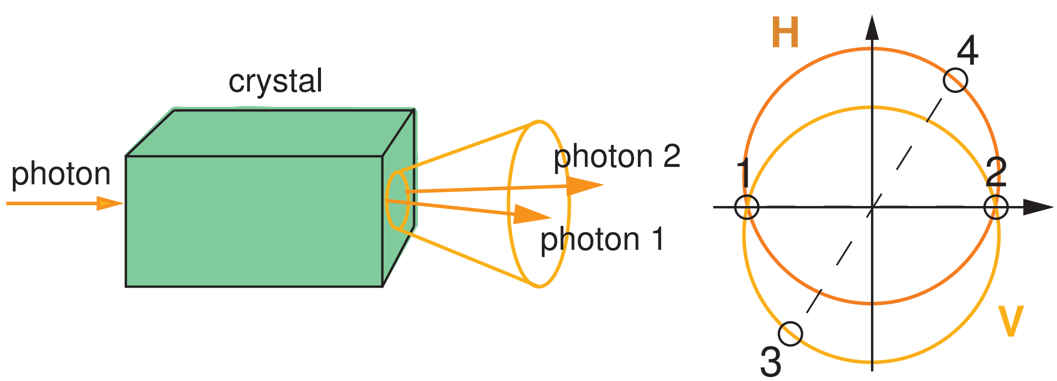
and circularly polarized photons with right-handed and left-handed orientation, respectively,

$$|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)\ , \quad |L\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle) \tag{5}$$

The two-photon states, when only polarization is taken into account, are vectors in the tensor product space spanned by the four vectors,

$$\begin{aligned} |HH\rangle &= |H\rangle \otimes |H\rangle\ , \quad |HV\rangle = |H\rangle \otimes |V\rangle\ , \\ |VH\rangle &= |V\rangle \otimes |H\rangle\ , \quad |VV\rangle = |V\rangle \otimes |V\rangle\ , \end{aligned} \tag{6}$$

(Note that even if the photons are bosons there is no symmetry constraint on the two-photon states, since we assume that the two photons can be distinguished by their different direction of propagation.)



As a specific way to produce entangled photon pairs we consider the method of *parametric down conversion*, as described below and sketched in the Figure 2 and 3.

As illustrated in Fig. 2a a (weak) beam of photons enter a crystal, where each photon due to the non-linear interaction with the crystal is split into two photons. These appear with equal energy, half the energy of the incoming photon. The transverse momentum of the emerging photons is fixed so that their direction of propagation is limited to a cone, as indicated in the figure. The photons appear with constant probability around the cone. However, due to conservation of total transverse momentum, the two photons in each a pair are correlated so that they always are emitted at opposite sides of the cone.

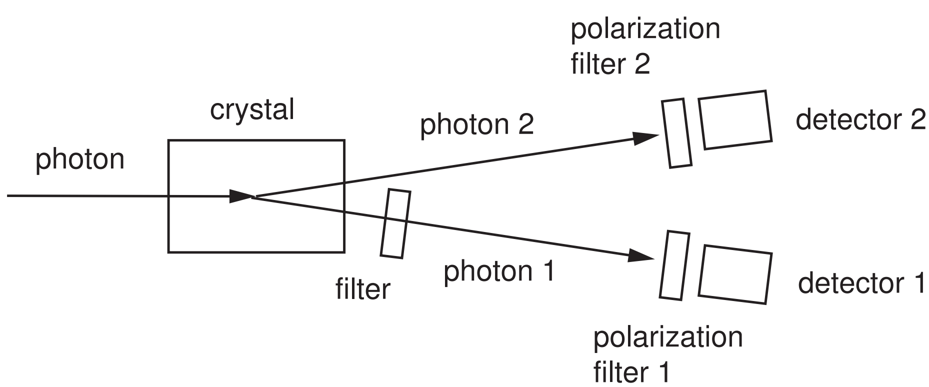
There is furthermore a polarization effect, since photons with horizontal and vertical polarization (relative to the crystal planes) do not propagate in exactly the same way. As a consequence the cones corresponding to these two polarizations are slightly shifted. This is shown in the head-on view of Fig. 2b, where the cone corresponding to polarization H is slightly lifted relative to the cone corresponding to polarization V.

Two photons in a correlated pair will be located on opposite points of the central point O, like the pair of points 1 and 2 and the pair 3 and 4, and they always appear with orthogonal polarization. As shown by the figure this means that for most directions of the emitted photons the polarization of each photon is uniquely determined by its direction of propagation. For such a pair the two-photon state is a product state of the form $|HV\rangle$. As an illustration, the pair $3, 4$ of directions of the cone, as shown in Fig.2b, will be of this type.

However two directions are unique since they lie on both cones. This is illustrated by the points 1 and 2 in Fig. 2b. A photon at one of these positions will be in a superposition of $|H\rangle$ and $|V\rangle$. Due to correlations between the photons a pair located at this position will be described by an entangled two-photon state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + e^{i\chi}|VH\rangle) \tag{7}$$

where the complex phase χ can be regulated in the experimental set up.



We assume in the following that a filter close to the crystal will single out photons only in the directions 1 and 2. This is schematically shown in Fig. 3. To analyze correlations between the two photons in each pair, polarization filters are applied to photons in both directions as also shown in the figure. Those that pass the polarization filters are registered in the detectors and the registrations are paired by use of coincidence counters.

The polarization filters may be represented by operators that project on linearly polarized states along a (rotated) direction

$$\hat{P}(\theta) = |\theta\rangle\langle\theta| \tag{8}$$

In the following we examine the expected results of the polarization measurements by calculating the following expectation values

$$
\begin{aligned}
P_1(\theta_1) &\equiv \left\langle \hat{P}_1(\theta_1) \right\rangle && \text{photon 1} \\
P_2(\theta_1) &\equiv \left\langle \hat{P}_2(\theta_2) \right\rangle && \text{photon 2} \\
P_{12}(\theta_1, \theta_2) &\equiv \left\langle \hat{P}_1(\theta_1)\hat{P}_2(\theta_2) \right\rangle && \text{photon 1 and photon 2}
\end{aligned} \tag{9}
$$

a) Assume a series of N entangled photon pairs are used in an experiment. In this series n_1 photons are registered in detector 1, n_2 photons are registered in detector 2 and and n_{12} are registered at coincidence in the two detectors. What are the relations between the registered frequences n_1/N, etc. and the the expectation values P_1, P_2 and P_{12}?

b) For the general two-photon state of the form (7) find the density operator of the two-photon pair, and find the corresponding reduced density operators for photon 1 and photon 2.

We consider three different situations where the the incoming photon pairs are in the state (7) with I: $\chi = \pi$, II: $\chi = 0$ and III: $\chi = \pi/2$.

c) Consider an input state of the form I. Determine the detection probability P_1 of photon 1 as a function of the angle θ_1 of polarizer 1. Do the same with P_2 for photon 2. Determine next the probability P_{12} for detecting photons at both analyzers as a function of the angles θ_1 and θ_2. What do the results tell about correlations of the two photons?

d) Consider next a two-photon input state of the form II. Examine the same questions as in c). Are the results obtained rotationally invariant? Compare the cases c) and d).

e) Consider finally the case III. Find also in this case the expectation values P_1, P_2 and P_{12} as fuctions of the angles of the polarizers. Show that in this case there exists a mixed state, which is an *incoherent* mixture of $|HV\rangle$ and $|VH\rangle$, that has identical expectation values.

f) The Bell inequality, which is based on an assumed set of ”hidden variables” as a source of the statistical distributions, can be written as a constraint on the function P_{12} in the following way (see Sect. 2.3.2 of the lecture notes),

$$
F(\theta_1, \theta_2, \theta_3) \equiv P_{12}(\theta_2, \theta_3) - |P_{12}(\theta_1, \theta_2) - P_{12}(\theta_1, \theta_3)| \geq 0 \tag{10}
$$

Examine the Bell inequality in the cases I, II and III for the special choice of angles $\theta_1 = 0$, $\theta_2 = \theta$ and $\theta_3 = 2\theta$ by plotting the function $F(0, \theta, 2\theta)$. Comment on which of the cases that show that the Bell inequality is not satisfied. Is there a relation between the conclusion for the case III and the results in e)?